

# POTENTIAL AUTOMORPHY FOR CERTAIN GALOIS REPRESENTATIONS TO $GL_{2n}$

THOMAS BARNET-LAMB

**ABSTRACT.** Building upon work of Clozel, Harris, Shepherd-Barron, and Taylor, this paper shows that certain Galois representations become automorphic after one makes a suitably large totally-real extension to the base field. The main innovation here is that the result applies to Galois representations to  $GL_{2n}$ , where previous work dealt with representations to  $GSp_n$ . The main technique is the consideration of the cohomology of the Dwork hypersurface, and in particular, of pieces of this cohomology other than the invariants under the natural group action.

The aim of this document is to prove a potential automorphy theorem: that is, a statement that certain Galois representations become automorphic when we make a large field extension. The overall strategy of the proof can appear slightly complicated, and before I commence with the details of the proof, I will begin with a fairly leisurely account of the considerations which lead to this overall strategy.

I will, however, first introduce just enough definitions to state the theorem which I will prove, and proceed to state it, in order that the reader not have to wade through the rest of the document in order to find it.

The first notion we will need to define is the notion of the *sign* of a polarizable Galois representation, after Bellaïche-Chenevier.

**Definition 1.** Let  $F$  be a CM field,  $l$  a rational prime, and  $r : \text{Gal}(\overline{F}/F) \rightarrow GL_n(\mathbb{Z}_l)$  a representation which is *polarizable*, in the sense that there exists an isomorphism  $r^c \cong r^\vee \epsilon_l^{1-n}$ . We can think of this as giving us a pairing  $\langle *, * \rangle$  on  $(\mathbb{Z}_l)^n$  satisfying  $\langle r(\sigma)v_1, r({}^c\sigma)v_2 \rangle = \epsilon_l^{1-n}(\sigma) \langle v_1, v_2 \rangle$  for each  $\sigma \in \text{Gal}(\overline{F}/F)$  and  $v_1, v_2 \in (\mathbb{Z}_l)^n$ . If  $r$  is in addition assumed to be absolutely irreducible, this pairing will either be symmetric or antisymmetric—and whether it is symmetric or antisymmetric turns out to only depend on  $r$ . We define the *sign* of  $r$  to be  $+1$  if the pairing is symmetric,  $-1$  if it is antisymmetric.

The point of this definition is that just as two dimensional Galois representations come in two kinds, odd and even, with radically different properties (odd representations are generally well-behaved and even representations are a mystery), there is a similar dichotomy for higher-dimensional representations. This is what is captured by the Bellaïche-Chenevier sign. Those representations with sign  $+1$  are the ‘good’ ones (generalizing odd two dimensional representations), and it will come as little surprise that we will have to restrict our theorems to such representations.

---

1991 *Mathematics Subject Classification.* 11R39 (primary), 11F23 (secondary).

*Key words and phrases.* Galois representation, potential automorphy, potential modularity, Dwork hypersurface.

The author was partially supported by NSF grant DMS-0600716 and by a Jean E. de Valpine Fellowship.

(The theorems of [4] contain such a restriction implicitly, since they deal only with symplectic Galois representations with totally odd multiplier, which will have sign +1 automatically.)

Since we restrict our attention to polarized representations of sign +1 *with  $\mathbb{Z}_l$  coefficients*, there is some further information that we can extract. (By a *polarized* representation, I mean a polarizable representation together with a specific choice of isomorphism  $r^c \cong r^\vee \epsilon_l^{1-n}$ .) The polarization gives us a symmetric pairing on the underlying vector space  $V$  of the representation, and we can reduce mod  $l$  to get a symmetric pairing on the  $\mathbb{F}_l$  vector space  $V \otimes \mathbb{F}_l$ . Such a pairing has an associated invariant called the *determinant*, which is a well defined element of  $\mathbb{F}_l^\times / (\mathbb{F}_l^\times)^2$ , the multiplicative group of elements of  $\mathbb{F}_l$  modulo squares. (If we choose a basis  $\{e_i\}$  of  $V \otimes \mathbb{F}_l$ , and represent the pairing by a matrix  $M$  such that  $M_{ij} = \langle e_i, e_j \rangle$ , then this determinant is just  $\det M$ , accounting for the name—but the invariant itself is of course independent of any choice of basis.)

Note that it is important to distinguish the determinant of  $r$ , which is a character of  $G_F$ , from the determinant of the pairing associated to the polarization of  $\bar{r}$ , which is an element of  $\mathbb{F}_l^\times / (\mathbb{F}_l^\times)^2$ .

**Definition 2.** Given a polarization on a representation  $r$  as above, the *determinant of the polarization* will refer to the determinant of the pairing associated to the polarization of  $\bar{r}$ . We will say the polarization *has square determinant* if this determinant is the identity element of  $\mathbb{F}_l^\times / (\mathbb{F}_l^\times)^2$ .

This determinant of the polarization will add a technical restriction to our theorem: we will only be able to prove a representation  $r$  potentially modular when the determinant of the polarization of  $r$  is a square. It is worth remarking that while the sign +1 restriction reflects a deep reality in Galois representations, the restriction on the determinant of the polarization appears to be a relatively shallow technical problem: for instance, the polarization determinant invariant becomes meaningless if we allow extension of the field of coefficients. Thus one might hope that this restriction might be removed in future work.

We are now in a position to state our main theorem.

**Theorem 3.** *For each pair of positive even integers  $n, N$  with  $N \geq n + 6$  we can find a constant  $C(n, N)$  and a quadratic extension  $F^*(n, N)$  of  $\mathbb{Q}(\mu_N)$  with the following property<sup>1</sup>:*

*Suppose that  $F/F_0$  is a Galois extension of CM fields containing  $\mu_N$ . Suppose that  $l > C(n, N)$  is a rational prime which is unramified in  $F$  and  $l \equiv 1 \pmod{N}$ . (So  $l$  automatically splits in  $\mathbb{Q}(\mu_N)$ .) Suppose in addition that  $l$  splits in  $F^*(n, N)$ . Let  $v_q$  be a prime of  $F$  above a rational prime  $q \neq l$  such that  $q \nmid N$ . Let  $\mathcal{L}$  be a finite set of primes of  $F$  not containing primes above  $lq$ .*

*Suppose that we are given a representation*

$$r : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_n(\mathbb{Z}_l)$$

*enjoying the following properties:*

- (1)  *$r$  ramifies only at finitely many primes.*
- (2)  *$r^c \cong r^\vee \epsilon_l^{1-n}$*

---

<sup>1</sup>The astute reader will note that since  $N \geq n + 6$ , we could consider the constant  $C$  as just depending on  $N$ , by taking an appropriate maximum over  $n$ . Nonetheless, I have chosen to emphasize  $n$ , which is in some sense much more important than  $N$ , by leaving it in the notation.

- (3)  $r$  has sign  $+1$ , in the sense of Bellaïche-Chenevier.
- (4) For each prime  $\mathfrak{w}|l$  of  $F$ ,  $r|_{\mathrm{Gal}(\overline{F}_{\mathfrak{w}}/F_{\mathfrak{w}})}$  is crystalline with Hodge-Tate numbers  $\{0, 1, \dots, n-1\}$
- (5)  $r$  is unramified at all the primes of  $\mathcal{L}$
- (6)  $(r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\mathrm{ss}}$  and  $\bar{r}|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})}$  are unramified, with  $(r|_{\mathrm{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\mathrm{ss}}$  having Frobenius eigenvalues  $1, (\#k(v_q)), \dots, (\#k(v_q))^{n-1}$
- (7)  $(\det \bar{r})^2 \cong \epsilon_l^{n(1-n)} \pmod{l}$
- (8)  $\bar{r}|_{\mathrm{Gal}(\overline{F}/F(\zeta_l))}$  is ‘big’.<sup>2</sup>
- (9)  $\overline{F}^{\ker \mathrm{ad} \bar{r}}$  does not contain  $F(\zeta_l)$
- (10)  $\bar{r}$  satisfies, for each prime  $\mathfrak{w}|l$  of  $F$ :

$$\bar{r}|_{I_{F_{\mathfrak{w}}}} \cong 1 \oplus \epsilon_l^{-1} \oplus \dots \oplus \epsilon_l^{1-n}$$

- (11)  $\bar{r}$  admits a polarization with determinant a square.

Then there is a CM field  $F'$  containing  $F$  which is Galois over  $F_0$  and linearly independent from  $\overline{F}^{\ker \bar{r}}$  over  $F$ . Moreover, all primes of  $\mathcal{L}$  and all primes of  $F$  above  $l$  are unramified in  $F'$ . Finally, there is a prime  $w_q$  of  $F'$  over  $v_q$  such that  $r|_{\mathrm{Gal}(\overline{F}/F')}$  is automorphic of weight 0 and type  $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$ .

Before I close the introduction, a few remarks are in order. This theorem generalizes work of Harris, Shepherd-Barron and Taylor. The key advances in this work are

- The representation  $r$  can now map into  $\mathrm{GL}_n$ ; in the earlier work, it was required to map into  $\mathrm{GSp}_n$ . (Restrictions were also placed on the multiplier.)
- The ability to vary the integer  $N$  is new. In the earlier work,  $n+1$  replaces  $N$  in all conditions above which refer to  $N$ , and no integer  $N$  is mentioned. This makes these conditions significantly more restrictive, for instance, the older theorem requires that  $l \equiv 1 \pmod{n+1}$ .

This paper relies heavily on work of Katz in [5] and on the lifting theorems of Clozel, Harris and Taylor in [1]. The question of looking at other parts of the cohomology of the Dwork hypersurface was raised by Guralnick, Harris and Katz in [3].

**Acknowledgements:** I would like to thank my advisor, Richard Taylor, for suggesting this problem to me and for immeasurable help in all aspects of my work on it.

## 1. THE STRATEGY

The basic approach of all potential automorphy proofs, following Taylor (see [9]), is the following. (This is a *very rough* sketch—many important details are omitted and there are even some deliberate lies!) The key ingredient in the proof is a family  $\mathfrak{F}$  of algebraic varieties whose cohomology is ‘very flexible’. In particular, given specified  $l$ -adic and  $l'$ -adic representations  $r$  and  $r'$ , and subject to certain conditions, we can find (over some suitably large totally real field) an element  $V$  of the family whose mod  $l$  cohomology looks like the residual representation of  $r$ , and whose mod  $l'$  cohomology looks like the residual representation of  $r'$ . We then apply this taking  $r$  to be the Galois representation which we would like to show

<sup>2</sup>Or more precisely, if we let  $r'$  denote the extension of  $r$  to a continuous homomorphism  $\mathrm{Gal}(\overline{F}/F^+) \rightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$  as described in section 1 of [1]; then  $\bar{r}'|_{\mathrm{Gal}(\overline{F}/F(\zeta_l))}$  is ‘big’.

is potentially modular and  $r'$  to be some Galois representation which is of a very special form—so special that we already know it to be modular. (For instance, say, we could take it to be induced from a character.) Over the field of definition of this variety (which may well be a very large extension of the field we started with)  $r$  and the cohomology of  $V$  (resp  $r'$  and the cohomology of  $V$ ) agree mod  $l$  (resp  $l'$ ).

We then apply a modularity lifting theorem twice. First, we argue that since the cohomology of  $V$  looks like  $r'$  (which is modular) mod  $l'$ , it must be modular; then we argue that since  $r$  looks like the cohomology of  $V$  (which is now known to be modular) mod  $l'$ , it too must be modular.

This strategy is summarized as Figure 1.

- |  |
|--|
| <ol style="list-style-type: none"> <li>1 We are given an <math>l</math>-adic Galois representation <math>r</math> which we want to prove modular.</li> <li>2 We construct a ‘nice’ <math>l'</math>-adic Galois representation <math>r'</math>.</li> <li>3 We find a variety <math>V</math> in our family whose cohomology looks like <math>r</math> mod <math>l</math> and <math>r'</math> mod <math>l'</math>. (This passes to a large field extension.)</li> <li>4 Applying a lifting theorem, we can deduce that the cohomology of <math>V</math> is modular.</li> <li>5 Applying a lifting theorem again, we can deduce that <math>r</math> is modular.</li> </ol> |
|--|

FIGURE 1. The naïve strategy

Alas, this simple schema turns out to be too naïve to work in practice at present. But before we consider the ways in which we must modify it to make a workable strategy, let us say a few words about how step 3 (perhaps the most ‘mysterious’ step given the short description above) can be accomplished. The basic method is to consider the moduli space of tuples  $(V, \iota, \iota')$  where  $V$  is an element of the family  $\mathfrak{F}$ ,  $\iota$  is an isomorphism between the mod  $l$  cohomology of  $V$  and  $r$  mod  $l$ , and  $\iota'$  is an isomorphism between the mod  $l$  cohomology of  $V$  and  $r'$  mod  $l'$ . We are essentially asking that this moduli space has a point over some sufficiently large totally real field. The theorem of Moret-Bailly allows us to do exactly that (subject to certain conditions), and even to control local behaviour, (including for example splitting and ramification) at certain primes. (This, we will see, will prove to be necessary for other reasons.)

Let us now turn to the inadequacies of the strategy above, and how to overcome them. One might guess that the two largest sources of trouble would be a) getting the details right in the argument which I just sketched for step 3, and b) facing up to the fact that lifting theorems have significant conditions attached, making their application a good deal more subtle than the blasé steps 4 and 5 above suggest. It turns out that nearly all of our woes will in fact arise from the second of these sources: perhaps rather miraculously, the details involved in fully working out step 3 do not necessitate any *strategic* changes to the argument. (They will, however, add some conditions to the final theorem.) The details of the lifting theorem applications, on the other hand, will necessitate several changes: one major, and a few which are more minor.

The major change comes from one particular condition which the lifting theorems have. Suppose that we are trying to apply a lifting theorem to deduce that  $r'$  is automorphic from the fact that it agrees mod  $l$  with some  $r$  which is automorphic. Given present technology, to apply the lifting theorem, we require the condition

that  $r'$  is discrete series at<sup>3</sup> some place  $v$ , and we also require that  $r$  and  $r''$  have the same type at  $v$ .

What does this mean for us? Let us first consider what changes must be made at step 5 in the argument. We need, for some place  $v$ , to control both the local representations of both  $r$  and the cohomology of  $V$  at  $v$ . As far as the first is concerned, our result will require an additional hypothesis: since  $r$  is given to us in the setup of the theorem, the only way we can control it is to add some condition to our theorem imposing what we need. However, having bitten that bullet (and once we have chosen exactly what condition we wish to impose), actually imposing the condition is a simple stroke of the pen. Controlling  $H(V)$ , however, will require ‘real work’: we need to have some handle on the family  $\mathfrak{F}$ .

It turns out that our control of the local representations occurring in elements of the family  $\mathfrak{F}$  is weak but not non-existent. In particular, given a place  $v_q$  where we want some control, there are certain conditions under which  $H(V)$  will look Steinberg at that place. These ‘certain conditions’ are (perhaps not suprisingly) local conditions at  $v_q$ . (That is,  $v_q$ -adic conditions on the parameters in the family). We recall that the theorem of Moret-Bailly allows us to impose such local conditions when we choose out  $V$  from the family.

This leads to the following plan for ‘fixing up’ step 5. We add a hypothesis to our theorem requiring that the representation  $r$  looks Steinberg at some place  $v_q$ . (Since there’s no other representation we can locally force  $H(V)$  to have, this has to be the representation we ask  $r$  to have.) We then impose a local condition when we choose  $V$  from our family to ensure that it too looks Steinberg at  $v_q$ . This revised strategy is shown in Figure 2.

- |   |
|---|
| <ol style="list-style-type: none"> <li>1 We are given an <math>l</math>-adic Galois representation <math>r</math> which we want to prove modular, which is Steinberg at some place <math>v_q</math>.</li> <li>2 We construct a ‘nice’ <math>l'</math>-adic Galois representation <math>r'</math>.</li> <li>3 We find a variety <math>V</math> in our family whose cohomology looks like <math>r \bmod l</math> and <math>r' \bmod l'</math>, and which is Steinberg at <math>v_q</math>. (This passes to a large field extension.)</li> <li>4 Applying a lifting theorem, we can deduce that the cohomology of <math>V</math> is modular.</li> <li>5 Applying a lifting theorem again, we can deduce that <math>r</math> is modular.</li> </ol> |
|---|

FIGURE 2. The strategy, taking into account the need for a Steinberg condition to apply the final lifting theorem

‘Fixing up’ step 4 will require a little more strategic modification of the argument. If we were to try to replicate the argument that we have just tried, we would be forced to try to gain control of the ‘nice’ Galois representation  $r'$  to try to arrange that it be Steinberg at  $v_q$ , to match the cohomology of  $V$ . Since we are thinking of the ‘nice’ representation as being induced from a character, this, of course, is a non-starter. Although  $r'$  may be discrete series in many places, there is no way that we can ask that it look Steinberg anywhere. But, on the other hand, the only real tool we have for proving complicated representations to be modular *ex nihilo*

---

<sup>3</sup>By *is discrete series at  $v$*  we mean that the restriction of  $r'$  to the local Galois group at  $v$  corresponds under local Langlands to a discrete series representation.

is that they be induced. We need some conduit which will allow us to transfer the modularity of the induced representation to the cohomology of  $V$ .

A very natural idea is to introduce an ‘intermediate’ representation  $r''$ . The idea is that it agrees with  $r'$  mod  $l'$ , and is Steinberg at  $v_q$  but agrees with  $r'$  in looking like some other discrete series representation at some other place. One would then use a lifting theorem twice: once to deduce the modularity of  $r''$  from that of  $r'$ , and then again to deduce the modularity of  $H(V)$  from that of  $r''$ . In each case, there is a suitable discrete series place to use as a pivot.

It turns out that it is possible to construct such a bridging representation using a Ramakrishnan-style lifting theorem, subject only to the condition that  $r'$  *looks* Steinberg mod  $l'$  at  $v_q$ . (It is clear that this condition is required, since otherwise the stipulations that  $r''$  look like  $r'$  mod  $l'$  and that it be Steinberg at  $v_q$  are incompatible.) And once we pass to a field extension which makes  $r'$  agree mod  $l'$  with  $H(V)$ , which is Steinberg at  $v_q$ , we will indeed satisfy this condition.<sup>4</sup>

This gives the modified strategy shown in Figure 3.

- 1 We are given an  $l$ -adic Galois representation  $r$  which we want to prove modular, which is Steinberg at some place  $v_q$ .
- 2 We construct a ‘nice’  $l'$ -adic Galois representation  $r'$ , which looks discrete series at some place  $p$ .
- 3 We find a variety  $V$  in our family whose cohomology looks like  $r$  mod  $l$  and  $r'$  mod  $l'$ , and which is Steinberg at  $v_q$ . (This passes to a large field extension.)
- 3a In particular,  $r'$  mod  $l'$  looks Steinberg at  $v_q$  mod  $l'$
- 4 We construct, using (3a), a  $l'$ -adic representation  $r''$  which a) agrees with  $r'$  at  $p$  b) agrees with  $r'$  mod  $l'$ , and c) is Steinberg at  $v_q$ .
- 5 Applying a lifting theorem and the fact that  $r'$  is modular, we can deduce that the  $r''$  is modular. (We ‘pivot’ at the place  $p$ , where they are both the same discrete series representation.)
- 6 Applying a lifting theorem and the fact that  $r''$  is modular, we can deduce that the cohomology of  $V$  is modular. (Pivot at  $v_q$ .)
- 7 Applying a lifting theorem again, we can deduce that  $r$  is modular. (Pivot at  $v_q$ .)

FIGURE 3. The strategy, now incorporating steps to ensure Steinberg conditions for both applications of lifting theorems

Before we proceed to discuss some of the more minor details that need to be considered in order to complete the strategy and allow us to move on to the details, it will be helpful to rearrange some of the steps in the above argument, in order to slightly reduce the number of things we need to keep in our heads at any one time. In particular, the result of the argument contained in steps 2, 4, 5 and 6 can be encapsulated into a lemma.

<sup>4</sup>Note that in an earlier preprint of this paper, the strategy was somewhat more complicated at this point. The root cause was that the lifting theorems of [1] required that certain powers of  $\#k(v_q)$  were distinct modulo  $l$ ; this meant that  $v_q$  could not be allowed to split in the field extension one makes to define  $V$ ; in turn, this meant that  $r'$  would have had to *already* look Steinberg mod  $l'$  at  $v_q$  to find the variety  $V$ . This is not possible to achieve in general, requiring the introduction of an auxiliary prime  $q'$  and significant additional complication in the argument.

(We will, in fact, be able to get away without going in to all the details involved in much of the argument which establishes the lemma, since we can cite results unchanged from the papers [1], [4].<sup>5</sup>)

Hiding the details involved in proving the lemma, gives a simplified strategy<sup>6</sup> shown in Figure 4.

- 1 We are given an  $l$ -adic Galois representation  $r$  which we want to prove modular, which is Steinberg at some place  $v_q$ .
- 2 *Lemma:* We can, given such  $r$ , find a mod  $l'$  representation  $r'$  such that any representation which is Steinberg at  $v_q$  and agrees with  $r' \bmod l'$  is modular.
- 3 We find a variety  $V$  in our family whose cohomology looks like  $r \bmod l$  and  $r' \bmod l'$ , and which is Steinberg at  $v_q$ . (Pass to field extension.)
- 4 Deduce that the cohomology of  $V$  is modular from the lemma.
- 5 Applying a lifting theorem, deduce that  $r$  is modular. (Pivot at  $v_q$ .)

FIGURE 4. The strategy simplified by having a lemma

The remaining modifications we must make to this general strategy are relatively small. Firstly, both the lifting theorem used in steps 5, and the ability to construct members of our family with certain cohomology (used in step 3) have, as the reader will probably have been aware, further conditions which we have been suppressing in our discussion. In some cases, these work out to be conditions on the representation  $r$  which we are trying to prove modular, and we will have to ‘pass through’ these conditions to the user of our theorem. The most notable condition of this form is that  $r$ ’s inertial representation at  $l$  must satisfy a rather restrictive condition to ensure that it ‘looks like’ the members of our family.

Secondly, it is useful to strengthen our theorem a little by allowing the user to control ramification at an auxiliary set of primes,  $\mathcal{L}$ . This is simply a matter of careful bookkeeping, and propagating the relevant data and conditions around the proof.

Finally, the situation regarding the family  $\mathfrak{F}$  is slightly more complicated than we have, thus far, been admitting. It is not in fact the case that we work directly with the whole cohomology of the varieties in our family. Rather, a particular abelian group acts on that cohomology, allowing us to decompose the cohomology according to the action of characters of that abelian group; it is within a particular piece of that decomposition that we will find the spaces that we want. Thus it is better to think of us working with a family of motives rather than a family of varieties *per se*.

In the next section, we define the family  $\mathfrak{F}$  and explain these things in more detail. We then prove a result which allows us to construct members of the family of motives with prescribed residual representations, and which are Steinberg at certain places, as required by steps 3 and 4’ of the strategy above. In the section

<sup>5</sup>Perhaps a little confusingly, the argument which establishes this fact is split between these two papers, with steps 4 and 6 being in the former paper and the remainder in the latter.

<sup>6</sup>In this strategy, the astute reader might be wondering *why*, exactly, the  $r'$  chosen in step 2 needs to depend on  $r$ , since its defining property apparently doesn’t depend on  $r$ . The point is that we are also choosing  $l'$  in this step, and *this* must be chosen depending on  $r$ : in particular,  $r$  must be unramified at  $l'$ .

after that, we prove the lemma required in step 2. In the final section, we put these results together to prove Theorem 3.

## 2. GEOMETRY

**2.1. The Dwork family.** Our aim in this section is to prove a proposition that allows us to find varieties with prescribed residual representations, and in order to do so we must introduce the Dwork family, within which we will find the varieties we seek. Let  $N$  be a positive integer. Fix a base ring  $R_0 = \mathbb{Z}[\frac{1}{N}, \mu_N]$ , where  $\mu_N$  denotes the  $N$ th roots of unity. We consider the scheme  $Y$ :

$$Y \subset \mathbb{P}^{N-1} \times \mathbb{P}^1$$

over  $R_0$  defined by the equations

$$\mu(X_1^N + X_2^N + \cdots + X_N^N) = N\lambda X_1 X_2 \cdots X_N$$

(using  $(X_1 : \cdots : X_N)$  and  $(\mu : \lambda)$  as coordinates on  $\mathbb{P}^{N-1}$  and  $\mathbb{P}^1$  respectively.) We consider  $Y$  as a family of schemes over  $\mathbb{P}^1$  by projection to the second factor. We will label points on this  $\mathbb{P}^1$  using the affine coordinate  $t = \lambda/\mu$ , and will write  $Y_t$  for the fiber of  $Y$  above  $t$ . (The notation broadly follows Katz's paper [5], except that I use  $N$  in place of his  $n$ ,  $Y$  for his  $\mathbb{X}$ , and the varieties I consider are less general than his—corresponding to the case  $W = (1, 1, \dots, 1)$  and  $d = n$  in his notation. In particular, our notation is *not* directly compatible with the notation of [4].)

There is a natural group acting on this family. Let  $\mu_N$  denote the  $N$ th roots of unity in  $R_0$ , and let  $\Gamma$  denote the  $N$  fold power  $(\mu_N)^N$ . Let  $\Gamma_W$  denote the subgroup of  $\Gamma$  consisting of all elements  $(\zeta_1, \dots, \zeta_N)$  with  $\prod_{i=1}^N \zeta_i = 1$  and let  $\Delta$  denote  $\mu_N$  embedded diagonally in  $\Gamma$ . Then the group  $\Gamma_W$  acts on  $Y$  with the element  $(\zeta_1, \dots, \zeta_N)$  acting via

$$((X_1 : \cdots : X_N), t) \mapsto ((\zeta_1 X_1 : \cdots : \zeta_N X_N), t)$$

The subgroup  $\Delta$  acts trivially.

The family  $Y$  is smooth over the open set  $U = \text{Spec } R_0[\lambda, \frac{1}{\lambda^N - 1}]$  away from the roots of unity. We will now construct certain sheaves on  $U$ . Let  $l$  be a prime number which splits in  $\mathbb{Q}(\mu_N)$ , and assume we have chosen an embedding  $\iota$  of  $R_0$  into  $\overline{\mathbb{Q}_l}$ . Let  $T_0^{(l)} = U[1/l]$ , and form lisse sheaves

$$(1) \quad \mathcal{F}_l^i := R^i \pi_* \overline{\mathbb{Q}_l}$$

$$(2) \quad \mathcal{F}^i[l] := R^i \pi_* \mathbb{Z}/l\mathbb{Z}$$

on  $T_0^{(l)}$ . (We will suppress the superscript  $(l)$  where it is clear from context.) Similarly, let  $M$  be an integer,  $T_0^{(M)} = U[1/M]$ , and define a lisse sheaf  $\mathcal{F}^i[M] := R^i \pi_* \mathbb{Z}/M\mathbb{Z}$  on  $T_0^{(M)}$ .

We are interested particularly in the sheaf  $\mathcal{F}_l^{N-2}|_{T_0}$ . From now on, we assume that  $N$  is even; then this  $\mathcal{F}_l^{N-2}|_{T_0}$  will contain a  $\mathbb{Q}_l(-N/2+1)$  as a direct summand (a power of the hyperplane class from the ambient  $\mathbb{P}$ ) with nonzero self-intersection under Poincaré duality. We will write  $\text{Prim}_l^{N-2}$  for the annihilator of this summand, so:

$$\mathcal{F}_l^{N-2}|_{T_0^{(l)}} = \text{Prim}^{N-2} \oplus \mathbb{Q}_l(-N/2+1)$$

As has been remarked,  $\Gamma_W/\Delta$  acts on our family, and respects the decomposition  $\mathcal{F}_l^{N-2}|_{T_0^{(l)}} = \text{Prim}^{N-2} \oplus \mathbb{Q}_l(-N/2+1)$ , and so acts on the sheaf  $\text{Prim}_l^{N-2}$  we have



just defined: thus we can decompose  $Prim_l^{N-2}$  into eigensheaves according to the characters of the group  $\Gamma_W/\Delta$ . Note that the coefficient ring of these sheaves will still be  $\mathbb{Q}_l$ , since  $l$  was chosen to split in  $\mathbb{Q}(\mu_N)$ .

The character group of  $\Gamma$  is  $(\mathbb{Z}/N\mathbb{Z})^N$ ; that of  $\Gamma_W$  is  $(\mathbb{Z}/N\mathbb{Z})^N/\langle W \rangle$  where we write  $W$  for the element  $(1, 1, \dots, 1)$ ; and the character group of  $\Gamma_W/\Delta$  is  $(\mathbb{Z}/N\mathbb{Z})_0^N/\langle W \rangle$  where we write  $(\mathbb{Z}/N\mathbb{Z})_0^N$  for  $\{(v_1, \dots, v_N) \in (\mathbb{Z}/N\mathbb{Z})^N \mid \sum_i v_i = 0\}$ . Thus the eigensheaves are labeled by elements of  $(\mathbb{Z}/N\mathbb{Z})_0^N/\langle W \rangle$ : we will write such an element as  $(v_1, \dots, v_N) \bmod W$  or simply as  $[(v_1, \dots, v_N)]$ , and shall write  $Prim_{l, [(v_1, \dots, v_N)]}^{N-2}$  for the piece of  $Prim_l^{N-2}$  where  $\Gamma_W/\Delta$  acts via  $[(v_1, \dots, v_N)]$ . Note that this labeling *depends on the choice of embedding*  $\iota : R_0 \rightarrow \overline{\mathbb{Q}}_l$ , since it requires us to have a preferred identification of the roots of unity in the coefficient ring  $\mathbb{Q}_l$  of the cohomology with the roots of unity in  $R_0$ .

We now are in a position to single out the particular piece of the cohomology with which we will work. From now on we will assume that we have another positive even integer  $n$  in mind, with  $N \geq n + 6$ . (This will be the dimension of the Galois representation which we will be working with in the end.) We will write  $k$  for  $n/2$ , and we will set

$$v = (0, \dots, 0, k+1, k+2, \dots, N/2-2, \\ N/2, N/2+1, \dots, N-k-2, N-k-1, N-1)$$

where we include every number once, except we *omit* the ranges  $1, \dots, k$  and  $N-k, \dots, N-2$  and the singleton  $N/2-1$ , and where the number of 0s at the beginning is  $n+1$ , calculated to ensure that there are  $N$  numbers in total. Note that these numbers add up to 0 mod  $N$ . Note also that the ranges above ‘make sense’ as long as  $N \geq n+4$ . For instance, if  $n=2, N=6$ , we take  $v = (0, 0, 0, 3, 4, 5)$ .

We will work with the piece  $Prim_{l, [v]}^{N-2}$  under which  $\Gamma_W/\Delta$  acts via this  $v \bmod W$ .<sup>7</sup> I will often write  $Prim_l$  for this sheaf, with the remaining data being understood. I will write  $Prim[l]$  for the corresponding sheaf constructed from  $\mathcal{F}^i[l]$ , and  $Prim[M]$  from the corresponding sheaf constructed from  $\mathcal{F}^i[M]$ .

**Proposition 4.** *We have the following facts about the varieties  $Y_t$  and the sheaves  $Prim[l]$ ,  $Prim_l$  and  $\mathcal{F}^i[l]$ . In the following,  $F$  will refer to a number field,  $v$  to a place of  $F$ . Recall that we are assuming  $l \equiv 1 \bmod N$  throughout.*

- (1) *If  $t \in T_0^{(l)}(F)$  and  $\mathfrak{q}$  is a place of  $F$  such that  $v_{\mathfrak{q}}(1-t^N) = 0$ , then  $Y_t$  has good reduction at  $\mathfrak{q}$ .*
- (2) *Suppose  $t \in T_0^{(l)}(F)$ . The Galois representation*

$$Prim_{l,t} : \text{Gal}(\overline{F}/F) \rightarrow GL_n(\mathbb{Q}_l)$$

*satisfies  $Prim_{l,t}^c \cong Prim_{l,t}^{\vee} \epsilon_l^{2-N}$ . Similarly  $Prim[l]_t^c \cong Prim[l]_t^{\vee} \epsilon_l^{2-N}$ . (Indeed these isomorphisms patch for different  $t$  to give a sheaf isomorphism.)*

- (3) *The sheaf  $Prim_l$  has rank  $n$ . There is a tuple  $\vec{h} = (h(\sigma))_{\sigma \in \text{Hom}(F, \overline{\mathbb{Q}}_l)}$ , such that the Hodge-Tate numbers of  $Prim_l$  at the embedding  $\sigma$  are  $\{h(\sigma), h(\sigma) + 1, \dots, h(\sigma) + n - 1\}$ .*
- (4) *Let  $\vec{h}$  continue to denote the tuple defined in the previous part. Suppose  $\mathfrak{w}|l$ , and let  $\sigma \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$  denote the corresponding embedding. Then*

<sup>7</sup>This is the point of where we part company from [4]; they work with  $N = n+1$  and the piece  $[(0, 0, \dots, 0)]$ .

- $Prim_{l,0}|_{I_w} \cong \epsilon_l^{-h(\sigma)} \oplus \epsilon_l^{-h(\sigma)-1} \oplus \cdots \oplus \epsilon_l^{1-h(\sigma)-n}$ , and  $Prim[l]_0|_{I_w} \cong \epsilon_l^{-h(\sigma)} \oplus \epsilon_l^{-h(\sigma)-1} \oplus \cdots \oplus \epsilon_l^{1-h(\sigma)-n}$
- (5) Let  $\mathfrak{q}$  be a prime of  $F$  above a rational prime which does not divide  $N$ . If  $\lambda_{\mathfrak{q}} \in T_0^{(l)}(F_{\mathfrak{q}})$  has  $v_{\mathfrak{q}}(\lambda_{\mathfrak{q}}) < 0$ , then  $(Prim_{l,\lambda_{\mathfrak{q}}})^{ss}$  is unramified, and  $(Prim_{l,\lambda_{\mathfrak{q}}})^{ss}(\text{Frob}_{\mathfrak{q}})$  has eigenvalues  $\{\alpha, \alpha \# k(\mathfrak{q}), \alpha(\#k(\mathfrak{q}))^2, \dots, \alpha(\#k(\mathfrak{q}))^{n-1}\}$  for some  $\alpha$ .
- (6) Let  $\mathfrak{q}$  be a prime of  $F$  above a rational prime which does not divide  $N$ . If  $\lambda_{\mathfrak{q}} \in T_0^{(l)}(F_{\mathfrak{q}})$  has  $v_{\mathfrak{q}}(\lambda_{\mathfrak{q}}) < 0$  and  $l|v_{\mathfrak{q}}(\lambda_{\mathfrak{q}})$ , then  $(Prim[l]_{\lambda_{\mathfrak{q}}})$  is unramified (even without semisimplification).
- (7) The monodromy of  $Prim_l$  is Zariski dense in  $\{A \in GL_n | \det A = \pm 1\}$ .

*Proof.* Point (1) is trivial. Point (2) comes from the fact that there is a perfect Poincaré duality pairing between  $Prim_{l,[v]}^{N-2} (=Prim_l)$  and  $Prim_{l,[-v]}^{N-2}$  towards  $\overline{\mathbb{Q}}_l(2-N)$ , and the fact that we can identify  $Prim_{l,[-v]}^{N-2}$  as the complex conjugate of  $Prim_{l,[v]}^{N-2}$ . (And then a similar argument for  $Prim[l]_t$ .)

It will prove useful to skip over points (3) and (4) and return to them later. To begin our analysis of points (5) and (6), let us note that it suffices, by an argument identical to that used to prove Lemma 1.15 of [4], to establish that, for  $\lambda_{\mathfrak{q}}$  of the form given, the monodromy of  $Prim_l$  around infinity is generated by a unipotent matrix with minimal polynomial  $(X-1)^n$ . Now we will apply Lemma 10.1 of [5]. It is clear from the definition of  $v$  we gave that point (4) of the equivalent conditions given in this lemma is true (viz, that the value 0 occurs more than once and no other value does); whence we can deduce the equivalent condition (2), which is the unipotence we need.

Next, we move to establish point (7). We apply Lemma 10.3 of [5]. It is immediate to see that the  $v$  we chose above does not have  $-v$  a permutation of  $v$ . Thus we are in case (1) of Lemma 10.3, and the geometric monodromy is dense in  $\{A \in GL_n | \det A = \pm 1\}$ , establishing point (7) of the present proposition.

We now move on to establish point (3). First, we will apply Lemma 3.1 of [5], which gives a recipe for computing the ranks of the eigensheaves of  $Prim_l^{N-2}$ , and another recipe for computing the Hodge-Tate numbers. We will apply the recipe for the ranks. We are asked to consider the coset of elements of  $(\mathbb{Z}/N\mathbb{Z})_0^N$  representing  $v$ , and in particular, those elements of the coset which are *totally nonzero*; that is, contain no 0s. The translate  $v - (y, y, \dots, y)$  will be totally nonzero iff  $y$  does not occur in  $v$ ; as discussed above, our  $v$  omits precisely  $n$  congruence classes mod  $N$ , hence there are  $n$  totally nonzero representatives. The rank equals the number of totally nonzero representatives, which will therefore be  $n$ .

Now, we apply Lemma 10.4 of [5], which tells us that (when the equivalent conditions of Lemma 10.1 of [5] hold, as they do for us) the Hodge-Tate weights form an unbroken string of ones; that is, the Hodge-Tate numbers are of exactly the form we require, where we define  $h(\sigma)$  to be the smallest Hodge-Tate number at the embedding  $\sigma$ .

For point (4), first observe that the group  $\Gamma/\Delta$  (rather than just  $\Gamma_W/\Delta$ ) actually acts on  $H(Y_0)$ , allowing us to decompose  $Prim_{l,0}$  further into eigensheaves for  $\Gamma/\Gamma_W$ . Proposition I.7.4 of [2] tells us that these eigensheaves are all one dimensional, and since  $l$  is chosen to split in  $\mathbb{Q}(\mu_N)$ , this tells us that  $Prim_{l,0}$  is a direct sum of characters, which are crystalline with Hodge-Tate numbers  $\{h, h+1, \dots, h+n-1\}$  by point (3). This establishes the first part; and this

also suffices to prove the second, since as  $l \equiv 1 \pmod{N}$ , we have  $l > N > n$  and the characters  $\epsilon_l^{-1}, \epsilon_l^{-2}, \dots, \epsilon_l^{1-n}$  have distinct reductions mod  $l$ .  $\square$

Now, as in [4], we can use Theorem 7.5 and Lemma 8.4 of [7] (or Theorem 5.1 of [8]), and deduce the following corollary from part (7) of the previous proposition:

**Corollary 5.** *There is a constant  $C(n, N)$  such that if  $M$  is an integer divisible only by primes  $p > C(n, N)$  and if  $t \in T_0^{(M)}$  then the map*

$$\pi_1(T_0^{(M)}, t) \rightarrow \mathrm{GL}_n(\mathrm{Prim}[M]_t)$$

*surjects onto  $\mathrm{SL}_n^\pm(\mathrm{Prim}[M]_t)$ . (Here  $\mathrm{SL}_n^\pm(\mathrm{Prim}[M]_t)$  denotes the group of automorphisms of  $\mathrm{Prim}[M]_t$  with determinant  $\pm 1$ .) (We may, and shall, additionally assume that  $C(n, N) > n$ .)*

**2.2. The determinant.** It will be important for us to study the determinant  $\det \mathrm{Prim}_l$ . Our main tool in doing so will be the main theorem of [5], which relates the sheaves  $\mathrm{Prim}_l$  to certain hypergeometric sheaves—so we will make a detour studying those. Let us write  $B$  for the scheme  $\mathbb{G}_m - \{1\}$  over  $R_0$ . Suppose we are given multisets  $S_\chi$  and  $S_\rho$  of characters  $\mu_N \rightarrow \mu_N$ , each of size  $k$ . (A quick point of convention: although we have been writing such characters ‘additively’ up until now, as elements of  $\mathbb{Z}/N\mathbb{Z}$ , it will be convenient in this section to switch to multiplicative notation to match the notation used by Katz. Thus, for instance, 1 will denote the trivial character.) In section 4 of [?], given these multisets, Katz defines a certain rank  $k$  lisse sheaf  $\mathcal{H}^{\mathrm{can}}(S_\chi, S_\rho)$  on  $B$ . We can consider this to be a representation of  $\pi_1(B)$ . Since  $B$  has a rational point (we will choose, in particular, the point  $2^N$ ), we can consider  $\pi_1(B)$  to be  $\pi_1(B \times \mathbb{Q}^{ac}) \rtimes G_{\mathbb{Q}(\mu_N)}$ . The determinant  $\det \mathcal{H}^{\mathrm{can}}(S_\chi, S_\rho)$  will be a character of this group, and any such character will factor through the abelianization

$$(\pi_1(B \times \mathbb{Q}^{ac}) \rtimes G_{\mathbb{Q}(\mu_N)})^{ab} = (\pi_1(B \times \mathbb{Q}^{ac})^{ab})_{G_{\mathbb{Q}(\mu_N)}} \times (G_{\mathbb{Q}(\mu_N)})^{ab}$$

Such a character will factor as a product of a character of  $(\pi_1(B \times \mathbb{Q}^{ac})^{ab})_{G_{\mathbb{Q}(\mu_N)}}$  and a character of  $(G_{\mathbb{Q}(\mu_N)})^{ab}$ . We will write  $\det \mathcal{H}^{\mathrm{can}}(S_\chi, S_\rho)|_{G_{\mathbb{Q}(\mu_N)}}$  for this character of  $G_{\mathbb{Q}(\mu_N)}$ . Finally, we note that if  $S_\chi = \{\chi\}$  and  $S_\rho = \{\rho\}$  have size 1,  $\mathcal{H}^{\mathrm{can}}(\{\chi\}, \{\rho\})$  is already a character, which we will call  $\lambda^{\mathrm{can}}(\{\chi\}, \{\rho\})$ . We can again factor this as a character of  $(\pi_1(B \times \mathbb{Q}^{ac})^{ab})_{G_{\mathbb{Q}(\mu_N)}}$  and a character of  $(G_{\mathbb{Q}(\mu_N)})^{ab}$ , and we can write  $\lambda_{G_{\mathbb{Q}(\mu_N)}}^{\mathrm{can}}(\{\chi\}, \{\rho\})$  for the latter character.

**Lemma 6.** *We have that*

$$\begin{aligned} & (\det \mathcal{H}^{\mathrm{can}}(\{\chi_1, \dots, \chi_n\}, \{1, \dots, 1\}))|_{G_{\mathbb{Q}(\mu_N)}}^2 \\ &= \lambda_{G_{\mathbb{Q}(\mu_N)}}^{\mathrm{can}}(\{\chi_1\}, \{1\})^{2n} \dots \lambda_{G_{\mathbb{Q}(\mu_N)}}^{\mathrm{can}}(\{\chi_n\}, \{1\})^{2n} \epsilon_l^{n(1-n)} \end{aligned}$$

*Proof.* Since both sides are characters which factor through  $(G_{\mathbb{Q}(\mu_N)})^{ab}$ , it will suffice by Chebotarev to show that they agree on Frobenii. But at a finite place  $\mathcal{P}$

above a rational place  $q$ , we have that

$$\begin{aligned}
(\det \mathcal{H}^{\text{can}}(S_\chi, S_\rho)|_{G_{\mathbb{Q}(\mu_N)}})^2(\text{Frob}_{\mathcal{P}}) &= \frac{(\det \mathcal{H}(S_\chi, S_\rho))^2(\text{Frob}_{\mathcal{P}})}{\left(\prod_i (-g(\psi, \chi_i)) \prod_j (-g(\bar{\psi}, \bar{\rho}_j))\right)^{n \deg \mathcal{P}}} \\
&= \left( \frac{\prod_i \chi_i((-1)^{n-1})^2 q^{n(n-1)} \prod_{i,j} (-g(\bar{\psi}, \bar{\rho}_j / \bar{\chi}_i))}{\left(\prod_i (-g(\psi, \chi_i)) \prod_j (-g(\bar{\psi}, \bar{\rho}_j))\right)^n} \right)^{\deg \mathcal{P}} \mathcal{L}_\Lambda(\text{Frob}_{\mathcal{P}})^2 \\
&= \left( \frac{q^{n(n-1)} \prod_{i,j} (-g(\bar{\psi}, \bar{\rho}_j / \bar{\chi}_i))}{\prod_i (-g(\psi, \chi_i))^n \prod_j (-g(\bar{\psi}, \bar{\rho}_j))^n} \right)^{\deg \mathcal{P}} \mathcal{L}_\Lambda(\text{Frob}_{\mathcal{P}})^2
\end{aligned}$$

(Here, on the first line we use the local definition of  $\mathcal{H}^{\text{can}}$  towards the bottom of page 10 of [5], together with the compatibility of the local and global definitions given at the bottom of page 11. On the second line we use the arithmetic determinant formula 8.12.2 of [6]. The character  $\Lambda := \prod_{\chi \in S_\chi} \chi$ )

Thus

$$\begin{aligned}
&(\det \mathcal{H}^{\text{can}}(\{\chi_1, \dots, \chi_n\}, \{1, \dots, 1\})|_{G_{\mathbb{Q}(\mu_N)}})^2(\text{Frob}_{\mathcal{P}}) \\
&= \left( \frac{q^{n(n-1)} (\prod_i (-g(\bar{\psi}, 1/\bar{\chi}_i)))^n}{\prod_i (-g(\psi, \chi_i))^n \prod_j (-g(\bar{\psi}, 1))^n} \right)^{\deg \mathcal{P}} \mathcal{L}_\Lambda(\text{Frob}_{\mathcal{P}})^2
\end{aligned}$$

On the other hand, it is easy to see by a similar argument that

$$(\lambda_{G_{\mathbb{Q}(\mu_N)}}^{\text{can}}(\{\chi_i\}, \{1\}))^2(\text{Frob}_{\mathcal{P}}) = \left( \frac{-g(\bar{\psi}, 1/\bar{\chi}_i)}{(-g(\psi, \chi_i))(-g(\bar{\psi}, 1))} \right)^{\deg \mathcal{P}} \mathcal{L}_{\chi_i}(\text{Frob}_{\mathcal{P}})^2$$

Whence

$$\begin{aligned}
\left( \frac{(\det \mathcal{H}^{\text{can}}(\{\chi_1, \dots, \chi_n\}, \{1, \dots, 1\})|_{G_{\mathbb{Q}(\mu_N)}})^2}{\prod_i (\lambda_{G_{\mathbb{Q}(\mu_N)}}^{\text{can}}(\{\chi_i\}, \{1\}))^{2n}} \right) (\text{Frob}_{\mathcal{P}}) &= q^{n(n-1) \deg \mathcal{P}} \left( \frac{\mathcal{L}_\Lambda}{\prod_i \mathcal{L}_{\chi_i}} \right) (\text{Frob}_{\mathcal{P}}) \\
&= q^{n(n-1) \deg \mathcal{P}} \\
&\quad (\text{as } \bigotimes_i \mathcal{L}_{\chi_i} \cong \mathcal{L}_{\prod_i \chi_i} = \mathcal{L}_\Lambda) \\
&= \epsilon_l^{n(1-n)} (\text{Frob}_{\mathcal{P}})
\end{aligned}$$

Whence we have the desired result.  $\square$

Next, there is a Galois character  $\Lambda_{v,W}$  defined in the Theorem 5.3 of [5]. We define a character  $G_{\mathbb{Q}(\mu_N)} \rightarrow \mathbb{Q}_l^\times$ :

$$\phi_l := \Lambda_{v,W} \prod_i (\lambda_{G_{\mathbb{Q}(\mu_N)}}^{\text{can}}(\{\chi_i\}, \{1\}))^2$$

(where the  $\chi_i$  are the maps  $\mu_N \rightarrow \mu_N$  naturally associated to the elements  $v_i \in \mathbb{Z}/N/\mathbb{Z}$ ) and we are now able to make the connection to the sheaves *Prim* which we have been studying.

**Lemma 7.** *We have that*

$$(\det \text{Prim}_{l,t=2})^2 = \phi_l^{2n} \epsilon_l^{n(1-n)}$$

*Proof.* We will apply the main theorem of [5], Theorem 5.3; in order to do so, we must first perform the procedure described at the beginning of section 5 of [5], which constructs certain lists of characters. In particular, we are meant to form the list  $List(-v, W)$ , which will in our case be a list of all the characters, with certain characters omitted, and extra copies of the trivial character added to pad the list to length  $N$ . (We see this by examining the choice of  $v$  made before Proposition 9; we must remember that although at that point we were writing 0 for the trivial character, here we will write 1 since we have switched to multiplicative notation for this section.) We are meant to cancel this against a list of all characters. The result will be the list

$$\text{Cancel}(List(\text{all } d), List(-v, W)) = (\{\text{certain characters}\}, \{1, 1, \dots, 1\})$$

The main Theorem 5.3 of [5] tells us that, as  $G_{\mathbb{Q}(\mu_N)}$  representations:

$$Prim_{l,t=2} = \mathcal{H}^{\text{can}}(\{\chi_1, \dots, \chi_n\}, \{1, 1, \dots, 1\})|_{G_{\mathbb{Q}(\mu_N)}} \otimes \Lambda_{v,W}$$

(recall that we chose the splitting of  $\pi_1(B)$  using the rational point  $2^N$  on  $B$ , which is the image of the point 2 under the  $N$ -th power map). Hence

$$\begin{aligned} (\det Prim_{l,t=2})^2 &= (\det \mathcal{H}^{\text{can}}(\{\chi_1, \dots, \chi_n\}, \{1, 1, \dots, 1\})|_{G_{\mathbb{Q}(\mu_N)}})^2 \Lambda_{v,W}^{2n} \\ &= \lambda_{G_{\mathbb{Q}(\mu_N)}}^{\text{can}}(\{\chi_1\}, \{1\})^{2n} \dots \lambda_{G_{\mathbb{Q}(\mu_N)}}^{\text{can}}(\{\chi_n\}, \{1\})^{2n} \Lambda_{v,W}^{2n} \epsilon_l^{n(1-n)} \\ &= \phi_l^{2n} \epsilon_l^{n(1-n)} \end{aligned}$$

which is what we wanted.  $\square$

Looking at the Hodge-Tate number of either side of the equation above at a prime  $\mathfrak{l}$  over  $l$ , and writing  $\text{HT}_{\mathfrak{l}}(\phi_l)$  for the Hodge-Tate number of  $\phi_l$  at that place, we get

$$\begin{aligned} 2 \times (\vec{h}(\mathfrak{l}) + (\vec{h}(\mathfrak{l}) + 1) + \dots + (\vec{h}(\mathfrak{l}) + n - 1)) &= 2n \text{HT}_{\mathfrak{l}}(\phi_l) + n(n - 1) \\ (2\vec{h}(\mathfrak{l}) + n - 1)n &= 2n \text{HT}_{\mathfrak{l}}(\phi_l) + n(n - 1) \\ (2\vec{h}(\mathfrak{l}))n &= 2n \text{HT}_{\mathfrak{l}}(\phi_l) \end{aligned}$$

and we deduce that  $\text{HT}_{\mathfrak{l}}(\phi_l) = \vec{h}(\mathfrak{l})$ . Thus we can use twisting by  $\phi_l$  to shift the Hodge-Tate numbers of an arbitrary representation by  $\vec{h}$ . We will write, given an  $l$ -adic representation  $r$ ,  $r(-\vec{h})$  for the twist of  $r$  by this character  $\phi_l$ , and  $r(\vec{h})$  for the twist by the inverse.

**2.3. A Galois descent.** We now need to prove a lemma which will play a small but critical role in the argument for the main theorem of this section. The reader may wish to skip these arguments at first reading, examine the proof of the main theorem at the end of the section, and having seen *why* we need the result we are about to prove, return to read the proof of it.

The issue it resolves is as follows. We have said that the basic structure of the argument which allows us to find prescribed residual representations in the cohomology of the Dwork family is the following: we construct a moduli space of points in the family which admit such isomorphisms, then we show it has a point over a suitable field by Moret-Bailly. The trouble is that we want to ensure that the point we construct will exist over a CM-field. Whereas Moret-Bailly lends itself well to constructing points over totally real fields (since this is expressible as a local

condition), asking for a CM field is not possible. Thus we need a more indirect approach.

The basic idea we will use is as follows. We will construct a scheme over a *totally real* field  $F^+$ , which will parametrize isomorphisms which exist when one passes to a certain quadratic totally imaginary extension  $F$  of that totally real field  $F^+$ . ( $F$  will, of course, be CM.) Moret-Bailly will allow us to show that this scheme has a point over a totally real extension field  $F^{+, '}$ —this will correspond to the isomorphism we need over a quadratic totally imaginary extension  $F'$  of  $F^{+, '}$ , which will be what we want.

Our goal is to prove a technical result which shows that a scheme parameterizing such isomorphisms does in fact exist.

Let us proceed to the actual setup. Suppose we have a base scheme  $S_0^+$ , defined over a totally real field  $F^+$  which contains the totally real subfield  $\mathbb{Q}(\mu_N)^+$  of  $\mathbb{Q}(\mu_N)$ . Let  $F := F^+(\mu_N)$  and let us write  $S_0$  for the base change  $S_0^+ \times_{F^+} F$ . Let  $\chi$  be a character of  $G_F$  into  $(\mathbb{Z}/M\mathbb{Z})^\times$ . Suppose further that we have two lisse rank  $n \bmod M$  sheaves  $\mathcal{A}, \mathcal{B}$  on the  $S_0$ . Suppose also that  $\mathcal{A}, \mathcal{B}$  satisfy  $\mathcal{A}^c \cong \mathcal{A}^\vee \otimes \chi$ ,  $\mathcal{B}^c \cong \mathcal{B}^\vee \otimes \chi$ , where  $\mathcal{A}^c$  is the ‘complex conjugate’ sheaf. (That is, the sheaf whose corresponding representation of  $\pi_1(S_0)$  is  $r \circ j_c$  where  $r$  is the representation of  $\pi_1(S_0)$  associated to  $\mathcal{A}$ , and  $j_c$  is the outer automorphism of  $\pi_1(S_0)$  coming from conjugation by a complex conjugation of the totally real subfield.)

Thinking of  $\mathcal{A}$  as a mod  $M$  representation  $V_{\mathcal{A}}$  of  $\pi_1(S_0)$ , this is the same as giving a pairing  $\langle *, * \rangle$  on  $V_{\mathcal{A}}$  which satisfies

$$\langle \sigma v_1, j_c(\sigma) v_2 \rangle = \chi(\sigma) \langle v_1, v_2 \rangle$$

and similarly for  $\mathcal{B}$ . We will suppose in addition that these pairings are *symmetric*. (That is,  $\mathcal{A}$  and  $\mathcal{B}$  have sign  $+1$  in the sense of Bellaïche-Chenevier.)

Suppose finally that there is an isomorphism  $\eta : (\wedge^n \mathcal{A})^{\otimes 2} \rightarrow (\wedge^n \mathcal{B})^{\otimes 2}$  and we have fixed one such isomorphism. This isomorphism should be compatible with the maps  $\mathcal{A}^c \cong \mathcal{A}^\vee \otimes \chi$ ,  $\mathcal{B}^c \cong \mathcal{B}^\vee \otimes \chi$  in the following sense. First note that  $\mathcal{A}^c \cong \mathcal{A}^\vee \otimes \chi$  will induce a map  $(\wedge^n \mathcal{A})(\wedge^n \mathcal{A})^c \rightarrow \chi^n$ , and hence we get (using a similar map for  $\mathcal{B}$ ) an distinguished isomorphism  $(\wedge^n \mathcal{A})(\wedge^n \mathcal{A})^c \cong (\wedge^n \mathcal{B})(\wedge^n \mathcal{B})^c$  (since both have specified isomorphisms to  $\chi^n$ ). We can take the tensor square to get an isomorphism  $(\wedge^n \mathcal{A})^{\otimes 2}(\wedge^n \mathcal{A})^{c \otimes 2} \cong (\wedge^n \mathcal{B})^{\otimes 2}(\wedge^n \mathcal{B})^{c \otimes 2}$ .  $\eta$  will also induce an isomorphism  $(\wedge^n \mathcal{A})^{\otimes 2}(\wedge^n \mathcal{A})^{c \otimes 2} \cong (\wedge^n \mathcal{B})^{\otimes 2}(\wedge^n \mathcal{B})^{c \otimes 2}$ ; we ask that these agree.

There is a certain important circumstance in which we can arrange for a compatible isomorphism  $\eta$  to exist. Suppose that we have *some* isomorphism  $\eta' : (\wedge^n \mathcal{A})^{\otimes 2} \rightarrow (\wedge^n \mathcal{B})^{\otimes 2}$ , and suppose moreover that there is some isomorphism  $\nu$  of vector spaces with pairing  $\mathcal{A} \rightarrow \mathcal{B}$ . (That is, this isomorphism  $\nu$  need not respect the Galois action at all, but does form a commutative square

$$\begin{array}{ccc} \mathcal{A}^c & \longrightarrow & \mathcal{A}^\vee \otimes \chi \\ \downarrow \nu & & \uparrow \nu^\vee \\ \mathcal{B}^c & \longrightarrow & \mathcal{B}^\vee \otimes \chi \end{array}$$

with the maps coming from our chosen isomorphisms  $\mathcal{A}^c \cong \mathcal{A}^\vee \otimes \chi$ ,  $\mathcal{B}^c \cong \mathcal{B}^\vee \otimes \chi$ .) Then taking the  $\wedge^n$  of  $\nu$  we can construct an isomorphism of 1-dimensional vector spaces (without Galois action)  $(\wedge^n \mathcal{A}) \rightarrow (\wedge^n \mathcal{B})$ , and hence (taking tensor squares), an isomorphism of 1-dimensional vector spaces (without Galois action)  $(\wedge^n \mathcal{A})^{\otimes 2} \rightarrow$

$(\wedge^n \mathcal{B})^{\otimes 2}$ . Now the key point: *given that  $\eta$  exists, this  $\eta'$  will automatically respect the Galois action.* (This is because the existence of  $\eta$  tells us that the characters by which Galois acts on each side are identical, which will force any isomorphism to be an isomorphism of 1D Galois modules.) It is also immediate, given the commutative diagram above in the construction of  $\eta'$ , that it is compatible with the isomorphisms  $\mathcal{A}^c \cong \mathcal{A}^\vee \otimes \chi$ ,  $\mathcal{B}^c \cong \mathcal{B}^\vee \otimes \chi$  in the sense we require.

Now, given a scheme  $R^+$  over  $S_0^+$  we can base change to form a scheme  $R := R^+ \times_{S_0^+} S_0$  over  $S_0$ . We can define a functor

$$S_{\mathcal{A}, \mathcal{B}} : \{S_0^+ \text{-schemes}\} \rightarrow \mathbf{Set}$$

$$R^+ \mapsto \left\{ \begin{array}{l} \text{Isomorphisms } \xi \text{ between the pull back to } R \\ \text{of } \mathcal{A} \text{ and } \mathcal{B}, \text{ such that } (\wedge^n \xi)^{\otimes 2} = \eta^2. \end{array} \right\}$$

(Here ‘isomorphisms’ means isomorphisms of sheaves *with stipulated pairings*  $\langle *, * \rangle$ .)

**Proposition 8.** *This functor is represented by a scheme.*

*Proof.* We will begin by constructing a certain finite étale cover  $S_1$  of the scheme  $S_0^+$ ; we will then show that this  $S_1$  represents the functor we want.

We can specify an finite étale cover of  $S_0$  by giving a representation of  $\pi_1(S_0^+)$  into the symmetric group on  $Q$  letters, where  $Q$  is the number of sheets, or equivalently by giving an action of  $\pi_1(S_0^+)$  on a  $Q$  element set. We can think of  $\mathcal{A}$  and  $\mathcal{B}$  as giving mod  $M$  representations of  $\pi_1(S_0)$ , say acting on the free  $\mathbb{Z}/M\mathbb{Z}$  modules  $V_{\mathcal{A}}$  and  $V_{\mathcal{B}}$  respectively. Thus we can immediately construct an étale cover of  $S_0$  by allowing  $\pi_1(S_0)$  to act on the finite set  $X$  of isomorphisms of vector spaces  $\iota : V_{\mathcal{A}} \mapsto V_{\mathcal{B}}$ , via the action  $A$  given by

$$(3) \quad \begin{aligned} \pi_1(S_0) \times X &\rightarrow X \\ (\alpha, \iota) &\mapsto \alpha^{-1} \iota \alpha \end{aligned}$$

(and indeed, it is easy to see that this corresponds to the variety  $\mathrm{Isom}(\mathcal{A}, \mathcal{B})$  over  $S_0$  parameterizing isomorphisms between  $\mathcal{A}$  and  $\mathcal{B}$  ignoring the pairing  $\langle *, * \rangle$ ). If we replaced the set  $X$  with the smaller set  $X_\eta$  of isomorphisms whose induced map on  $(\wedge^n)^{\otimes 2}$ 's is  $(\eta)^2$ , then we would get the variety parameterizing isomorphisms lifting  $\eta$ .

We wish, however, to construct an étale covering of  $S_0^+$ , which means we need to extend the above action to an action of  $\pi_1(S_0^+)$ . Now, if we write  $c$  for complex conjugation  $c \in \pi_1(S_0^+)$ , then  $\pi_1(S_0^+)$  is generated by  $c$  and  $\pi_1(S_0)$ ; so we just need to define an action of  $c$  on  $X_\eta$  which commutes in the right way with all the other actions we have defined.

Given an isomorphism  $\iota : V_{\mathcal{A}} \mapsto V_{\mathcal{B}}$ , we can define an isomorphism  $\tilde{\iota}$  as follows: for all  $v_1, v_2 \in V_{\mathcal{A}}$ , we impose  $\langle \iota v_1, \tilde{\iota} v_2 \rangle = \langle v_1, v_2 \rangle$ . (Thus  $\tilde{\iota}$  is the ‘inverse of the adjoint’ of  $\iota$ .) We can easily calculate that  $\tilde{\tilde{\iota}} = \iota$ , since:

$$\begin{aligned} \langle \tilde{\iota} v_1, \tilde{\tilde{\iota}} v_2 \rangle &= \langle v_1, v_2 \rangle = \mathrm{sgn} V_{\mathcal{A}} \langle v_2, v_1 \rangle = \mathrm{sgn} V_{\mathcal{A}} \langle \iota v_2, \tilde{\iota} v_1 \rangle \\ &= \mathrm{sgn} V_{\mathcal{A}} \mathrm{sgn} V_{\mathcal{B}} \langle \tilde{\iota} v_1, \iota v_2 \rangle = \langle \tilde{\iota} v_1, \iota v_2 \rangle \end{aligned}$$

(note that at this point we use the fact that both  $\mathcal{A}$  and  $\mathcal{B}$  have sign  $+1$ ; or, more precisely, that they have the same sign). Moreover, we note that for  $\alpha \in \pi_1(S_0)$ , we have  $(\alpha^{-1} \iota \alpha)^{\sim} = j_c(\alpha)^{-1} \tilde{\iota} j_c(\alpha)$  where  $j_c(\alpha)$  as above denotes conjugation by

complex conjugation; the demonstration goes as follows:

$$\begin{aligned} \langle \alpha^{-1} \iota \alpha v_1, j_c(\alpha)^{-1} \tilde{\iota} j_c(\alpha) v_2 \rangle &= \chi(\alpha^{-1}) \langle \iota \alpha v_1, \tilde{\iota} j_c(\alpha) v_2 \rangle \\ &= \chi(\alpha^{-1}) \langle \alpha v_1, j_c(\alpha) v_2 \rangle \\ &= \chi(\alpha^{-1}) \chi(\alpha) \langle v_1, v_2 \rangle = \langle v_1, v_2 \rangle \end{aligned}$$

These two relations ensure that we can extend our action  $A$  on  $X_\eta$  to an action of  $\pi_1(S_0^+)$  by stipulating that  $A(c)(\iota) = \tilde{\iota}$ . (The fact that this action preserves  $X_\eta$  inside  $X$  is a consequence of the fact that we chose the isomorphism  $\eta$  compatibly with the pairings on  $\mathcal{A}, \mathcal{B}$ .) Hence we have constructed an étale cover of  $S_0^+$ , which we will call  $S_\eta^+$ .

We now pass to consider the question of what it means to give a  $S_\eta^+$ -scheme, say  $f : R^+ \rightarrow S_\eta^+$ . From general facts about étale covers, this is the same as giving a  $S_0^+$ -scheme  $f_0 : R^+ \rightarrow S_0^+$  together with a point in  $X_\eta$  which is stabilized by the image of  $\pi_1(R^+)$  in  $\pi_1(S_0^+)$  under the map on  $\pi_1$  induced by  $f_0$ . Now, given such a map  $f_0$ , pullback induces a map  $R \rightarrow S_0$  and we will have a commutative diagram:

$$\begin{array}{ccccc} \pi_1(R^+) & \xrightarrow{f_0} & \pi_1(S_0^+) & \longrightarrow & \text{Sym}(X_\eta) \\ \uparrow \text{index 2} & & \uparrow \text{index 2} & \nearrow & \\ \pi_1(R) & \longrightarrow & \pi_1(S_0) & & \end{array}$$

To give a point in  $X_\eta$  stabilized by the image of  $f_0$  is to give

- (1) A point in  $X_\eta$  stabilized by the image of  $\pi_1(R)$  in  $\pi_1(S_0)$ ...
- (2) ...which is also fixed by  $c \in \pi_1(S_0^+)$ .

Now, point 1 here is equivalent (by e.g. the remarks immediately after equation 3) to giving an isomorphism  $\theta$  between the pullbacks of  $\mathcal{A}$  and  $\mathcal{B}$  from  $S_0$  to  $R$  ignoring the pairing  $\langle *, * \rangle$ . Then point 2 imposes additionally that  $\tilde{\theta} = \theta$ ; unpacking this, it is seen to be equivalent to  $\theta$  preserving the pairing  $\langle *, * \rangle$ . This is as required.  $\square$

One final remark should be made in this connection. What does it mean to give a point of the scheme (or equivalently the functor) just defined over a field  $K$  which *contains*  $\mathbb{Q}(\mu_N)$ ? A fairly easy check shows that this is just the same as giving an isomorphism between the pullback to  $K$  of  $\mathcal{A}$  and the pullback to  $K$  of  $\mathcal{B}$ , now *disregarding* the pairing.

**2.4. Realizing residual representations.** Let us note that the isomorphism  $\text{Prim}_{l,t}^c \cong \text{Prim}_{l,t}^\vee \epsilon_l^{2-N}$  from Proposition ?? makes  $\text{Prim}_{l,t}$  into a polarized Galois representation with sign  $+1$ . (Since  $N$  is even, the Poincaré duality pairing is symmetric and the multiplier,  $\epsilon_l^{2-N}$  is also even.) Thus we can consider the determinant of this polarization.

We are now in a position to prove a result allowing us to realize residual Galois representations in the cohomology of the family  $Y_t$ .

**Proposition 9.** *The family  $Y_t$  and the piece of its cohomology corresponding to  $\text{Prim}_{l,t}$  have the following property:*

*Suppose  $K/F$  is a Galois extension of CM fields, with totally real subfields  $K^+, F^+$ ,  $n$  is a positive integer,  $l_1, l_2 \dots l_r$  are distinct primes which are unramified in  $K$ , and that we are given residual representations*

$$\bar{\rho}_i : \text{Gal}(\bar{F}/F) \rightarrow \text{GL}_n(\mathbb{F}_{l_i}).$$



Suppose further that we are given  $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ , distinct primes of  $F$  above rational primes  $q_1, \dots, q_s$  respectively, and  $\mathcal{L}$  a set of primes of  $F$  not including the  $\mathfrak{q}_j$  or any primes above the  $l_i$ . Suppose that each  $\mathfrak{q}_j$  satisfies  $\mathfrak{q}_j \nmid N$ . Finally, suppose that the following conditions are satisfied for each  $i$ :

- (1)  $l_i > C(n, N)$
- (2)  $l_i \equiv 1 \pmod{N}$
- (3)  $\bar{\rho}_i$  is unramified at each prime of  $\mathcal{L}$  and at the  $l_k$  for  $k \neq i$ .
- (4) For each prime  $\mathfrak{w}$  above  $l_i$ , we have that

$$\bar{\rho}_i|_{I_{\mathfrak{w}}} \cong 1 \oplus \epsilon_{l_i}^{-1} \oplus \dots \oplus \epsilon_{l_i}^{1-n}$$

- (5) We have that there exists a polarization  $\bar{\rho}_i^c \cong \bar{\rho}_i^\vee \epsilon_{l_i}^{1-n}$ ; given this, we can associate to  $\bar{\rho}_i$  a sign in the sense of Bellaïche-Chenevier and we require that this sign is  $+1$ . We also require that the polarization can be chosen so that its determinant is the same as the determinant of the polarization  $\mathrm{Prim}_{l_i, t}^c \cong \mathrm{Prim}_{l_i, t}^\vee \epsilon_{l_i}^{2-N}$ . Finally, we require that  $(\det \bar{\rho}_i)^2 \cong \epsilon_{l_i}^{n(1-n)}$

Then we can find a CM field  $K'/F$ , linearly disjoint from  $K/F$ , a finite-order character  $\chi_i : \mathrm{Gal}(\bar{\mathbb{Q}}/K') \rightarrow \mathbb{Q}_{l_i}$  for each  $i$ , and a  $t \in K'$  such that,

- (1) All primes of  $F$  above the  $\{l_i\}_{i=1, \dots, r}$  and all the  $\mathcal{L}$  are unramified in  $K'$
- (2) For all  $i$ ,  $Y_t$  has good reduction at each prime above lying above  $l_i$ , and each prime above the primes of  $\mathcal{L}$ .
- (3) For all  $i$  and  $\mathfrak{w}|l_i$ ,  $\mathrm{Prim}_{\mathfrak{w}, t}(\vec{h}) \otimes \chi_i$  is crystalline with H-T numbers  $\{0, 1, \dots, n-1\}$ .
- (4) For each  $\mathfrak{Q}$  above some  $\mathfrak{q}_j$ , we have that  $(\mathrm{Prim}_{l_i, t})^{ss}$  and  $\chi_i$  are unramified at  $\mathfrak{Q}$ , with  $(\mathrm{Prim}_{l_i, t}^{ss}(\vec{h}) \otimes \chi)(\mathrm{Frob}_{\mathfrak{Q}})$  having eigenvalues  $\{1, \#k(\mathfrak{Q}), \#k(\mathfrak{Q})^2, \dots, \#k(\mathfrak{Q})^{n-1}\}$ .
- (5)  $\mathrm{Prim}[l_i]_t(\vec{h}) \otimes \bar{\chi}_i = \bar{\rho}_i$  for all  $i$ .

*Proof.* Throughout this proof, we will set  $M = \prod l_i$ .

Since  $\mathrm{GL}_n(\mathbb{Z}/M\mathbb{Z})$  is just  $\mathrm{GL}_n(\mathbb{Z}/l_1\mathbb{Z}) \times \dots \times \mathrm{GL}_n(\mathbb{Z}/l_r\mathbb{Z})$ , we can combine the  $\rho_i$  into a single representation

$$\rho_{\mathbb{Z}/M\mathbb{Z}} : \mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{GL}_n(\mathbb{Z}/M\mathbb{Z})$$

and similarly we can combine the  $\phi_l$ 's mod  $l$  for different  $l_i$  too, to get a mod  $M$  character; we will write  $(\vec{h})$  for the twist by this character also.

We note that, thinking of  $\mathrm{Prim}[M]$  and  $\bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}$  as  $\mathbb{Z}/M\mathbb{Z}$  modules with pairing and Galois action, they are certainly isomorphic once we disregard the Galois action and only keep the pairing. Since  $\mathbb{F}_l$  vector spaces with pairing are classified by the determinant of the pairing, and since  $\bar{\rho}_i$  and  $\mathrm{Prim}[l_i]$  have polarizations with the same determinant, by hypothesis 5, this is immediate.

Next, we must study the determinant  $\det \mathrm{Prim}$ , a representation of  $\pi_1(T_0^{(M)})$ . Using the rational point 2 on  $T_0^{(M)}$ , we write (as we did above)  $\pi_1(T_0^{(M)})$  as  $\pi_1(T_0^{(M)} \times \mathbb{Q}^{ac}) \rtimes G_{\mathbb{Q}(\mu_N)}$  and observe that any character of  $\pi_1(T_0^{(M)})$  factors through

$$(\pi_1(T_0^{(M)} \times \mathbb{Q}^{ac})^{ab})_{G_{\mathbb{Q}(\mu_N)}} \times (G_{\mathbb{Q}(\mu_N)})^{ab}$$

and we can write  $\det \mathrm{Prim}$  as the product of two characters,  $\det \mathrm{Prim} = \psi_1 \psi_2$ , where  $\psi_1$  factors through  $(\pi_1(T_0^{(M)} \times \mathbb{Q}^{ac})^{ab})_{G_{\mathbb{Q}(\mu_N)}}$  and  $\psi_2$  through  $G_{\mathbb{Q}(\mu_N)}$ .

Now,  $\psi_1$  maps into the image of geometric monodromy, which we know to be  $\pm 1$ , since we know that geometric monodromy acts on  $\mathrm{Prim}$  via matrices with determinant  $\pm 1$ . Thus  $(\psi_1)^2$  is trivial. And  $\psi_2$  was studied above in Lemma

7. We deduce that  $(\det \text{Prim}[M](\vec{h}))^2 = (\phi^{-n} \det \text{Prim}[M])^2 = \phi^{-2n} \psi_1^2 \psi_2^2 = \phi^{-2n} (\det \text{Prim}[M]_{t=2})^2 = \epsilon_l^{n(1-n)}$

On the other hand, by hypothesis, we have that  $(\det \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}})^2 = \epsilon_l^{n(1-n)}$ . Thus  $(\det \text{Prim}[M])^2 = (\det \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}})^2$ , and we may fix a choice of isomorphism  $\eta : (\det \text{Prim}[M](\vec{h}))^{\otimes 2} \rightarrow (\det \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}})^2$ . (Indeed, we can choose that this isomorphism be compatible with the polarizations on  $\text{Prim}[M]$  and  $\bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}$ , in the sense defined in the previous section. As was discussed there, to prove that this is possible it will suffice to give an isomorphism  $\bar{\rho}_{\mathbb{Z}/M\mathbb{Z}} \rightarrow \text{Prim}[M]$  as vector spaces with pairing but without Galois action, as was done above.)

These preliminaries done, we are now on to the heart of the proof. The basic method is to consider the moduli space of tuples  $(Y_t, \iota)$  where  $Y_t$  is an element of the family  $\mathfrak{F}$ , and  $\iota$  is an isomorphism between  $\rho_{\mathbb{Z}/M\mathbb{Z}}$  and the mod  $M$  cohomology of  $Y_t$  twisted by the character  $\bar{\phi}_{\mathbb{Z}/M\mathbb{Z}}$ . We shall show that this has a point over a large totally real field using the theorem of Moret-Bailly.

Let us proceed with the details. It will be useful to give a name to the totally real analogue of our base space  $T_0^{(M)}$ ; so let us define  $R_0^+$  to be  $\mathbb{Z}[\mu_N, \frac{1}{N}]^+$  and  $T_0^{(M)+}$  to be  $\text{Spec } R_0^+[\frac{1}{\lambda^N - 1}, \lambda, \frac{1}{M}]$ . Now, let  $\mathcal{W}$  be a free  $\mathbb{Z}/M\mathbb{Z}$ -module of rank  $n$  with a continuous action of  $\text{Gal}(\bar{F}/F)$ ; we can think of this as a lisse étale sheaf on  $\text{Spec } F$ . In particular, we will be taking  $\mathcal{W}$  to be the module coming from  $\bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}(-\vec{h})$ . Given a  $T_0^{(M)+} \times_{\mathbb{Z}[\mu_N]^+} \text{Spec } F^+$  scheme  $S^+$ , we can pull back along

$$T_0^{(M)} \times_{\mathbb{Z}[\mu_N]} \text{Spec } F \rightarrow T_0^{(M)+} \times_{\mathbb{Z}[\mu_N]^+} \text{Spec } F^+$$

to get a  $T_0^{(M)} \times_{\mathbb{Z}[\mu_N]} \text{Spec } F$  scheme  $S$ , and we can consider isomorphisms between the pullback of  $\mathcal{W}$  to  $S$  and the pullback of  $\text{Prim}[M]$  to  $S$ .

Consider the functor  $T_{\mathcal{W}}$ :

$$\begin{aligned} \{T_0^{(M)+} \times_{\mathbb{Z}[\mu_N]^+} \text{Spec } F^+ \text{-schemes}\} &\rightarrow \mathbf{Set} \\ S^+ &\mapsto \left\{ \begin{array}{l} \text{Isomorphisms } \xi \text{ between the pull back} \\ \text{to } S \text{ of } \mathcal{W} \text{ and of } \text{Prim}[M] \text{ such that} \\ \text{the induced isomorphism } (\det \xi)^{\otimes 2} : \\ (\det \text{Prim}[M])^{\otimes 2} \rightarrow (\det \mathcal{W})^{\otimes 2} \text{ agrees with} \\ \eta. \end{array} \right\} \end{aligned}$$

This functor is represented by a scheme, which we will also denote by  $T_{\mathcal{W}}$ . (To see this, we simply apply Proposition 8.)

We then have the following facts:

- (1) *The scheme  $T_{\mathcal{W}}$  is geometrically connected.* To see this, we must see that the geometric monodromy acts transitively on the points in a fiber of  $T_{\mathcal{W}} \rightarrow T_0$ . This fiber is the set of isomorphisms between the rank  $n$   $\mathbb{Z}/M\mathbb{Z}$  modules  $\text{Prim}[M]$  and  $\mathcal{W}$  which preserve the determinant squared; any such isomorphism can be transformed into any other by the action of  $\{x \in \text{GL}_n(\mathbb{Z}/M\mathbb{Z}) \mid (\det x)^2 = 1\}$ . But we are then done by Corollary 5.
- (2) *If we let  $S_1$  denote the set of infinite places, and define*

$$\Omega_w = T_{\mathcal{W}}^{(M)}(F_w^+)$$

(where  $w$  refers to an infinite place) then these sets are nonempty. We claim that this has a point over  $0 \in T_0^{(M)}$ . To give such a point is to give an isomorphism between the pullbacks of  $\text{Prim}[M]_0$  and  $\mathcal{W}$  to  $\mathbb{R} \otimes_{\mathbb{F}^+} F$ ;

that is, to  $\mathbb{C}$ . But once we pull back to  $\mathbb{C}$ , all Galois action information is discarded, and all that remains are spaces with a pairing—and we saw these to be isomorphic at the beginning of the proof.

- (3) *If we let  $S_2$  denote the set of primes above the  $l_i$  together with the primes of  $\mathcal{L}$ , and define, for  $\mathfrak{w} \in S_2$*

$$\Omega_{\mathfrak{w}} = \{t^* \in T_0^{(M)}(F_{\mathfrak{w}}^{+nr}) \text{ above } t \in T_{\mathcal{W}}^{(M)}(F_{\mathfrak{w}}^{+nr}) \text{ s.t. } v_{\mathfrak{w}}(1 + t^N) < 0\}$$

*then these sets are nonempty.*

To see that these sets are isomorphic, we will actually show that there is a point in the sets above lying above the point  $0 \in T_0^{(M)}$ ; that is, we will show that the Galois representations  $\mathrm{Prim}[M]_0$  and  $\mathcal{W}$  become isomorphic once restricted to the absolute Galois group of  $(F_{\mathfrak{w}}^+)^{nr}$ ; or, in other words, once restricted to inertia. To see this, first use condition (3), which gives us what we require at  $\mathcal{L}$ . (Both representations are unramified, so trivial on inertia) Then use condition (4) at the places above the  $l_i$ , which tells us that the inertial representation of  $\mathcal{W} = \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}(-\vec{h})$  at a prime  $\mathfrak{w}$  above  $l_i$  is a direct sum of increasing powers of the cyclotomic character, starting with the  $h(\sigma)$ 'th power, where  $\sigma : F \rightarrow \overline{\mathbb{Q}}_l$  is the embedding corresponding to  $\mathfrak{w}$ ; and condition (2) together with conclusion (4) of Proposition 4 which tells us that  $\mathrm{Prim}_{l,0}$  takes exactly the same form.

- (4) *If we let  $S_3$  denote the set of the  $\mathfrak{q}_j$ , and define*

$$\Omega_{\mathfrak{q}_j} = \{t^* \in T_{\mathcal{W}}^{(M)}(\bar{F}_{\mathfrak{q}_j}) \text{ above } t \in T_0^{(M)}(\bar{F}_{\mathfrak{q}_j}) \text{ s.t. } v_{\mathfrak{q}_j}(t) < 0\}$$

*then these sets are nonempty.*

Again, we will show that there is a point in the set lying above the point  $0 \in T_0^{(M)}$ . This is immediate, since the mod  $M$  representations  $\mathcal{W}$  and  $\mathrm{Prim}[M]$  have finite image, and once we trivialize both by making a large local extension, they are isomorphic.

Thus, by the theorem of Moret-Bailly, in the version given as Proposition 2.1 of [4], we can find a field  $K'^+/F^+$ , disjoint from  $K^+/F^+$ , and a point  $t^* \in T_{\mathcal{W}}(K')$  (where  $K' := K'^+F$ ) lying above a point  $t$  in  $T_0(K')$  such that:

- All primes of  $S_2$  (that is, all the primes above the primes  $l_i$  and the primes of  $\mathcal{L}$ ) are unramified in  $K'$ . *Thus we get conclusion (1).*
- All primes of  $S_1$  split completely in  $K'$ . *Thus we conclude that  $K'^+$  is totally real and hence  $K'$  is CM.*
- For each  $j$ , we have  $t \in \Omega_{\mathfrak{q}_j}$ ; that is, for each  $j$  and for each prime  $\mathfrak{Q}$  above  $\mathfrak{q}_j$ , we have that  $v_{\mathfrak{Q}}(t) < 0$ . Thus, by part (5) of Proposition 4, we can conclude for each  $i$  that  $(\mathrm{Prim}_{l_i,t})^{ss}$  is unramified at  $\mathfrak{Q}$  and  $(\mathrm{Prim}_{l_i,t})^{ss}$  has  $\mathrm{Frob}_{\mathfrak{Q}}$  eigenvalues  $\{\beta_{i,\mathfrak{Q}}, \beta_{i,\mathfrak{Q}} \# k(\mathfrak{Q}), \beta_{i,\mathfrak{Q}} (\# k(\mathfrak{Q}))^2, \dots, \beta_{i,\mathfrak{Q}} (\# k(\mathfrak{Q}))^{n-1}\}$  for some  $\beta_{i,\mathfrak{Q}}$ . Making a further totally-real field extension unramified at the  $l_i$ , we can assume that, for each  $i$ , all the  $\beta_{i,\mathfrak{Q}}$  are 1 mod  $l_i$ .

We can then choose a character  $\chi_i : \mathrm{Gal}(\overline{\mathbb{Q}}/K') \rightarrow \overline{\mathbb{Q}}_{l_i}$  for each  $i$  lifting  $\bar{\chi}_i$  which is unramified at the primes of  $\mathcal{L}$ , the primes above the  $l_i$ , and the  $\mathfrak{Q}$  and which takes  $\mathrm{Frob}_{\mathfrak{Q}}$  to  $\beta_{i,j}^{-1}$ .

Then it is immediate that  $(\mathrm{Prim}_{l_i,t}^{ss}(\vec{h}) \otimes \chi_i)(\mathrm{Frob}_{\mathfrak{Q}})$  has eigenvalues  $\{1, \# k(\mathfrak{Q}), (\# k(\mathfrak{Q}))^2, \dots, (\# k(\mathfrak{Q}))^{n-1}\}$ . *Thus we get conclusion (4).*

- We have, for each prime  $\mathfrak{w}$  above either some  $l_i$  or some element of  $\mathcal{L}$ , that  $t \in \Omega_{\mathfrak{w}}$ ; that is,  $v_{\mathfrak{w}}(1 - t^N) < 0$ . Thus, by part 1 of Proposition 4,  $Y_t$  has

good reduction at  $\mathfrak{w}$  and  $\text{Prim}_{\mathfrak{w},t}$  is crystalline. The Hodge-Tate numbers are  $\{h(\sigma), h(\sigma) + 1, \dots, h(\sigma) + n - 1\}$  by part 3 of Proposition 4, where  $\sigma : F \rightarrow \overline{\mathbb{Q}}_l$  is the embedding corresponding to  $\mathfrak{w}$ . Thus  $\text{Prim}_{\mathfrak{w},t}(\vec{h}) \otimes \chi_i$  is crystalline with Hodge-Tate numbers  $\{0, \dots, n - 1\}$ . (Recall  $\chi_i$  is finite order and unramified at the the  $l_i$ .) *This gives us conclusions (2) and (3) of the present proposition.*

Finally, by definition of  $T_W$ , the point  $t^*$  gives us a specified isomorphism between  $\chi_{\mathbb{Z}/M\mathbb{Z}}^{-1} \otimes \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}(-\vec{h})$  and  $\text{Prim}[M]_t$ ; that is, we have

$$\text{Prim}[M]_t(\vec{h}) \otimes \bar{\chi}_{\mathbb{Z}/M\mathbb{Z}} = \bar{\rho}_{\mathbb{Z}/M\mathbb{Z}}$$

which is the final conclusion (5) of the present proposition. This concludes the proof.  $\square$

We close this section with a short argument showing that  $\text{Prim}[l]_0$ 's natural polarization coming from Poincare duality will have determinant a square for the  $l$  splitting in a certain quadratic extension of  $\mathbb{Q}(\mu_N)$

**Proposition 10.** *Suppose  $N$  is a positive integer; then there is a quadratic extension  $F^*(n, N)$  of  $\mathbb{Q}(\mu_N)$  such that for any  $l$  splitting in  $F^*(n, N)$ , the natural polarization on  $\text{Prim}[l]_0$  has determinant a square.*

*Proof.* Choose an arbitrary infinite place of  $\mathbb{Q}(\mu_N)$ , and consider  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathbb{Z})$ , the singular cohomology of the Fermat hypersurface  $Y_0$  with integral coefficients. We can extend coefficients to  $\mathcal{O}_{\mathbb{Q}(\mu_N)}$ , getting  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})$ , which will break up into eigenspaces under the action of the group  $\Gamma_W/\Delta$ . Let  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})_v$  denote the eigenspace corresponding to  $v$ . This will have a perfect integral Poincare duality pairing with  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})_{(-v)}$ , which is the complex conjugate of  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})_v$ ; combining Poincare duality with complex conjugation, we get a perfect integral pairing on  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})_v$  itself, which will have a determinant, a well-defined element  $\alpha$  of  $\mathcal{O}_{\mathbb{Q}(\mu_N)}$ . Let  $F^*(n, N) = \mathbb{Q}(\mu_N, \sqrt{\alpha})$ .

Now, the determinant of the Poincare duality pairing on  $\text{Prim}[l]_0$  is the same as the determinant of the pairing on  $H_{\text{ét}}(Y_0 \times \mathbb{C}, \mathbb{Z}_l)$ , (passing to the infinite place we chose discards the Galois action but leaves the pairing unaffected). This is, by the comparison theorem, the same as the determinant of the pairing on  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathbb{Z}_l)$ , which will be  $\alpha$ , considered as an element of  $\mathbb{Z}_l$ . (Recall  $\alpha$  was the determinant of the pairing on  $H_{\text{sing}}(Y_0 \times \mathbb{C}, \mathcal{O}_{\mathbb{Q}(\mu_N)})$ .) If  $l$  splits in  $F^*(n, N)$ , then  $\alpha \bmod l$  is a square in  $\mathbb{F}_l$ , and hence we are done.  $\square$

### 3. CONSTRUCTING A ‘SEED’ GALOIS REPRESENTATION

In our proof strategy above, we had as step 2 the establishment of the following roughly-stated lemma:

*Lemma:* We can, given such  $r$ , find a mod  $l'$  representation  $r'$  such that any representation which is Steinberg at  $v_q$  and agrees with  $r'$  mod  $l'$  is modular.

Our aim in this section is to prove this lemma. As mentioned above, in our statement of the strategy many conditions have been suppressed, and before we can proceed to the proof of the lemma we will need to state it more precisely, including all necessary conditions. Here is the precise statement of the lemma:

**Proposition 11.** *Suppose that  $F$  is a CM field,  $n$  and  $N$  are positive even integers,  $l$  is a prime which is unramified in  $F$ , and that we are given a representation*

$$r : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{SL}_n(\mathbb{Z}_l)$$

*Suppose further that  $v_q$  is a prime of  $F$  above a rational prime  $q \neq l$  and  $\mathcal{L}$  be a finite set of primes of  $F$  not containing primes above  $lq$ . Then we can find a rational prime  $l'$  and a mod  $l'$  representation*

$$\bar{r}' : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GSp}_n(\mathbb{F}_{l'})$$

*with multiplier  $\epsilon_l^{1-n}$ , which satisfy the following conditions:*

- (1)  $l' > C(n, N)$ ,  $l' \equiv 1 \pmod{4N}$ , and  $l'$  splits in  $F^*(n, N)$ . (Recall that the constant  $C(n, N)$  was defined in Corollary 5.)
- (2)  $\bar{r}'$  unramified at all primes of  $\mathcal{L}$  and above  $l$ .
- (3) For each prime  $\mathfrak{w}$  of  $F$  above  $l'$ , we have that

$$\bar{r}'|_{\mathrm{Gal}(\overline{F}_{\mathfrak{w}}/F_0)} \cong 1 \oplus \epsilon_{l'}^{-1} \oplus \cdots \oplus \epsilon_{l'}^{1-n}$$

- (4)  $\bar{r}$  unramified at  $l'$ .
- (5) Whenever  $F'/F$  is a field extension and  $r'' : \mathrm{Gal}(\overline{F}/F') \rightarrow \mathrm{GL}_n(\mathbb{Z}_{l'})$  is a  $l'$ -adic Galois representation which satisfies the following conditions:
  - (a) We have that  $r'' \cong (r'|_{\mathrm{Gal}(\overline{F}/F')}) \pmod{l'}$ .
  - (b)  $r''^c \cong r''^{\vee} \epsilon_l^{1-n}$
  - (c)  $r''$  ramifies at only finitely many primes
  - (d) For all places  $v|l$  of  $F$ ,  $r''|_{\mathrm{Gal}(\overline{F}_v/F_v)}$  is crystalline.
  - (e) For all  $\tau \in \mathrm{Hom}(F, \overline{\mathbb{Q}}_l)$  above a prime  $v|l$  of  $F$ ,

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r'' \otimes_{\tau, F_v} B_{\mathrm{DR}})^{\mathrm{Gal}(\overline{F}_v/F_v)} = \begin{cases} 0 & (i = 0, 1, \dots, n-1) \\ 1 & (\text{otherwise}) \end{cases}$$

- (f) For some prime  $\Omega$  above  $v_q$ , we have that  $r''|_{\mathrm{Gal}(\overline{F}_{\Omega}/F_{\Omega})}$  is unramified, with  $r''|_{\mathrm{Gal}(\overline{F}_{\Omega}/F_{\Omega})}(\mathrm{Frob}_{\Omega})$  eigenvalues  $\{\#k(\Omega)^j : j = 0, \dots, n-1\}$  (for some  $\alpha \in \overline{\mathbb{Q}}_l^{\times}$ ).

*then  $r''$  is automorphic over  $F$  of weight 0 and type  $\{\mathrm{Sp}_n(1)\}_{\{\Omega\}}$ .*

Before we start the proof of this proposition, I will give a few remarks to explain roughly where the extra conclusions which have now appeared in the statement come from. Point 5 is the conclusion from the informal statement: it says that agreeing with  $\bar{r}' \pmod{l'}$  allows us to deduce modularity (now subject to some more conditions (b)–(f), which are pretty standard extra conditions coming from the modularity theorems we will apply in proving Proposition 11).

The other conclusions are required so that we will be able to meet the hypotheses of Proposition 9 when we apply it. (In particular, 3 is the inertia condition we noted above.) For more details of exactly where these conditions fit into the jigsaw-puzzle of the overall argument, I refer the reader to the synoptic proof of Theorem 12 in the next section, and in particular to Figure 6.

*Proof of Proposition 11.* As mentioned above, we are lucky in that the argument we need is entirely contained in the earlier work [4] and [10]. (We are however slightly unlucky in that the results we need are split between these two papers). The facts we need from [10] are in a readily-citable form, but the arguments we need from [4] are not, being part of a longer argument (roughly speaking, they are

the first three pages, pp 22–25, in the proof of Theorem 3.1). We will therefore briefly describe exactly what we need to take from [4] and then go on to cite the results we need from the other paper.

We begin following the argument at the beginning of Theorem 3.1 of [4], taking  $r = 1$ ,  $n_1 = n$ , (indeed, from now on we will often without further comment write  $X$  where [4] writes  $X_1$ , for symbols  $X$ ), and  $F_0 = F$  (all other notation being the same). Choose  $E, M, \phi, l', \tilde{M}, \tilde{w}_{l'}, w_{l'}$  as in [4] (except that when we choose  $l'$ , we make sure that it splits in  $F^*(n, N)$ , as we trivially may). Construct  $\psi_{l'}$  as given by the recipe in the displayed equation on page 24, and use this to construct the character  $\bar{\theta}$  with the properties in the middle of page 24. Finally, construct  $I(\bar{\theta})$ .

We have now taken all we require from [4].  $I(\bar{\theta})$  is the representation  $r'$  we are seeking. (It has multiplier  $\epsilon_{l'}^{1-n}$  from the first bullet point on page 24.) Point 1 comes from the first two bullet points in the second set of bullet points on page 23 (and the fact that  $l'$  splits in a field containing  $\zeta_N$ ); and point 4 comes from the fourth bullet there. Points 2 and 3 comes from the first three bullet points concerning  $\bar{\theta}$  on page 24.

Now we will prove part 5; this is where we appeal to [10]. Suppose that we are given such a representation  $r''$ . We will show  $r''$  automorphic by appeal to Theorem 5.6 of [10] Conditions (1), (2), (3), (4), and (5) of that theorem are met by points (a-e) respectively. Condition (6) is immediate from point (f).  $\square$

#### 4. PUTTING THE PIECES TOGETHER

We are now in a position to use the various pieces we have accumulated to prove the main Theorem 3. The strategy is briefly recalled in Figure 5.

- 1 We are given an  $l$ -adic Galois representation  $r$  which we want to prove modular, which is Steinberg at some place  $v_q$ .
- 2 *Lemma:* We can, given such  $r$ , find a mod  $l'$  representation  $r'$  such that any representation which is Steinberg at  $v_q$  and agrees with  $r'$  mod  $l'$  is modular.
- 3 We find a variety  $V$  in our family whose cohomology looks like  $r$  mod  $l$  and  $r'$  mod  $l'$ , and which is Steinberg at  $v_q$ . (Pass to field extension.)
- 4 Deduce that the cohomology of  $V$  is modular from the lemma.
- 5 Applying a lifting theorem, deduce that  $r$  is modular. (Pivot at  $v_q$ .)

FIGURE 5. A reminder of the strategy

Let us also remind ourselves of the precise statement of the theorem. In the statement at the beginning of this paper, I tried to group the conditions in a way that will be of maximum use to users of the theorem. It may not be clear at all how the theorem, in that form, is related the strategy above. The restatement here will regroup the conditions in an attempt make the connection to our earlier discussion clearer; the reader should have little difficulty in convincing themselves that the two theorems are the same. The reader will see that the conditions have been grouped into three collections A, B and C; an explanation of the reasons behind this grouping will soon be given!

**Theorem 12** (Restatement of Theorem 3). *Suppose that  $F/F_0$  is a Galois extension of CM fields,  $n$  is a positive even integer,  $N \geq n + 6$  is a positive even integer*

such that  $F$  contains  $\mu_N$ ,  $l$  is a prime which is unramified in  $F$ , and that we are given a representation

$$r : \text{Gal}(\overline{F}/F) \rightarrow GL_n(\mathbb{Z}_l)$$

Suppose further that  $v_q$  is a prime of  $F$  above a rational prime  $q \neq l$  and  $\mathcal{L}$  be a finite set of primes of  $F$  not containing primes above  $lq$ , and that the following conditions are satisfied:

- A:**  $(r|_{\text{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\text{ss}}$  is unramified and  $(r|_{\text{Gal}(\overline{F}_{v_q}/F_{v_q})})^{\text{ss}}$  has Frobenius eigenvalues  $1, (\#k(v_q)), \dots, (\#k(v_q))^{n-1}$
- B1:**  $r^c \cong r^\vee \epsilon_l^{1-n}$ , with sign  $+1$ , and with some choice of polarization having determinant a square
- B2:**  $r$  ramifies only at finitely many primes.
- B3:** For each prime  $w|l$  of  $F$ ,  $r|_{\text{Gal}(\overline{F}_w/F_w)}$  is crystalline with Hodge-Tate numbers  $\{0, 1, \dots, n-1\}$ .
- B4:**  $\overline{F}^{\ker \text{ad} \bar{r}}$  does not contain  $F(\zeta_l)$
- B5:** Let  $\bar{r}$  denote the semisimplification of the reduction of  $r$ ; then  $\bar{r}(\text{Gal}(\overline{F}/F(\zeta_l)))$  is ‘big’ in the sense of ‘big image’.
- C1:** We have that  $q \nmid N$
- C2:**  $l > C(n, N)$ . (This constant was defined in Corollary 5.)
- C3:**  $l \equiv 1 \pmod{N}$ , and  $l$  splits in the extension  $F^*(N, n)$
- C4:**  $r$  is unramified at all the primes of  $\mathcal{L}$
- C5:** We have that:

$$\bar{r}|_{I_{F_w}} \cong 1 \oplus \epsilon_l^{-1} \oplus \dots \oplus \epsilon_l^{1-n}$$

- C6:** We have  $(\det \bar{r})^2 \cong \epsilon_l^{n(1-n)} \pmod{l}$

Then there is a CM field  $F'$  containing  $F$  which is Galois over  $F_0$  and linearly independent from  $\overline{F}^{\ker \bar{r}}$  over  $F$ . Moreover, all primes of  $\mathcal{L}$  and all primes of  $F$  above  $l$  are unramified in  $F'$ . Finally, there is a prime  $w_q$  of  $F'$  over  $v_q$  such that  $r|_{\text{Gal}(\overline{F}/F')}$  is automorphic of weight 0 and type  $\{\text{Sp}_n(1)\}_{\{w_q\}}$ .

Before we go any further, I will briefly remark on where the conditions here come from, hopefully relating the theorem as we have now restated it to our earlier discussion in Section 1. Condition A is the Steinberg hypothesis which should be familiar from the proof strategy. Conditions B1–5 are standard lifting theorem hypotheses which we have suppressed to far: since the strategy ends by applying a lifting theorem these hypotheses should come as no surprise. Conditions C1–5 are the extra conditions ‘passed up’ from Proposition 9. In particular, condition C5 is the inertial condition which I mentioned at the end of Section 1.

I will also reproduce the lifting theorem which I need to apply from [1]: we have to refer constantly to the conditions of this theorem, and so it is convenient to have a statement of the theorem to hand.

**Theorem 13** (Theorem 5.2 of [10]). *Let  $F$  be an imaginary CM field and let  $F^+$  be its maximal totally real subfield. Let  $n \in \mathbb{Z}_{\geq 1}$  and let  $l > n$  be a prime which is unramified in  $F$ . Let*

$$r : \text{Gal}(\overline{F}/F) \rightarrow GL_n(\overline{\mathbb{Q}}_l)$$

*be a continuous irreducible representation with the following properties. Let  $\bar{r}$  denote the semisimplification of the reduction of  $r$ . Suppose that:*

- (1)  $r^c \cong r^\vee \epsilon_l^{1-n}$

- (2)  $r$  is unramified at all but finitely many primes.  
(3) For all places  $v|l$  of  $F$ ,  $r|_{\text{Gal}(\overline{F}_v/F_v)}$  is crystalline.  
(4) There is an element  $a \in (\mathbb{Z}^n)^{\text{Hom}(F, \overline{\mathbb{Q}}_l)}$  such that
- for all  $\tau \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$  we have either  $l-1-n \geq a_{\tau,1} \geq \dots \geq a_{\tau,n} \geq 0$  or  $l-1-n \geq a_{\tau c,1} \geq \dots \geq a_{\tau c,n} \geq 0$
  - for all  $\tau \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$  and all  $i = 1, \dots, n$  we have  $a_{\tau c,i} = -a_{\tau, n+1-i}$
  - for all  $\tau \in \text{Hom}(F, \overline{\mathbb{Q}}_l)$  above a prime  $v|l$  of  $F$

$$\dim_{\overline{\mathbb{Q}}_l} \text{gr}^i(r \otimes_{\tau, F_v} B_{\text{DR}})^{\text{Gal}(\overline{F}_v/F_v)} = \begin{cases} 0 & i = a_{\tau,j} + n - j \text{ (for some } j) \\ 1 & \text{(otherwise)} \end{cases}$$

- (5) Let  $r_l$  denote the local Langlands correspondance, normalized as in Proposition 4.3.1 of [1], and  $||$  denote the modulus character. There is a non-empty finite set  $S$  of places of  $F$  not dividing  $l$  and for each  $v \in S$  a square integrable representation  $\rho_v$  of  $GL_n(F_v)$  over  $\overline{\mathbb{Q}}_l$  such that

$$(r|_{\text{Gal}(\overline{F}_v/F_v)})^{\text{ss}} = r_l(\rho_v)^{\vee}(1-n)^{\text{ss}}$$

If  $\rho_v = \text{Sp}_{m_v}(\rho'_v)$  then set

$$\tilde{r}_v = r_l((\rho'_v)^{\vee} ||^{(n/m_v-1)(1-m_v)/2})$$

Note that  $r|_{\text{Gal}(\overline{F}_v/F_v)}$  has a unique filtration  $\text{Fil}_v^j$  such that

$$\text{gr}_v^j r|_{\text{Gal}(\overline{F}_v/F_v)} \cong \tilde{r}_v \epsilon^j$$

for  $j = 0, \dots, m_v - 1$  and equals (0) otherwise. We assume that  $\tilde{r}_v$  has irreducible reduction  $\bar{r}_v$ . Then  $\bar{r}|_{\text{Gal}(\overline{F}_v/F_v)}$  inherits a filtration  $\bar{\text{Fil}}_v^j$  with

$$\text{gr}_v^j \bar{r}|_{\text{Gal}(\overline{F}_v/F_v)} \cong \bar{r}_v \epsilon^j$$

for  $j = 0, \dots, m_v - 1$ .

- (6)  $\overline{F}^{\ker \text{ad} \bar{r}}$  does not contain  $F(\zeta_l)$   
(7) Let  $r'$  denote the extension of  $r$  to a continuous homomorphism  $\text{Gal}(\overline{F}/F) \rightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_l)$ , where  $\mathcal{G}_n$  is the group defined at the beginning of [1]; then  $\bar{r}'(\text{Gal}(\overline{F}/F(\zeta_l)))$  is ‘big’.<sup>8</sup>  
(8) The representation  $\bar{r}$  is irreducible and automorphic of weight  $a$  and type  $\{\rho_v\}_{v \in S}$  with  $S \neq \emptyset$

*Proof of Theorem 12.* This is now a simple matter of combining the results we have accumulated according to our original strategy. (Note that the numbering of the steps here does not correspond directly to the numbering in the strategy.) Figure 6 may be of some help in understanding how the parts of the proof fit together.

*Step 1:* Given an  $r$  as in the theorem, we can immediately apply Proposition 11, constructing a rational prime  $l'$  and an  $l'$ -adic representation  $r'$ , satisfying the conclusions 1–5.

*Step 2:* We now apply Proposition 9 taking  $s = 1$ ,  $\mathfrak{q}_1 = v_q$  and  $r = 2, l_1 = l, l_2 = l'$ , and  $K' = \overline{K}^{\ker \bar{r}}$ ; and using  $\bar{\rho}_1 = \bar{r}$  and  $\bar{\rho}_2 = \bar{r}'$  (the semisimplification of the reduction of  $r'$ ). Conditions 1, 2, 3, 4 on  $\bar{\rho}_1 = \bar{r}$  and  $l_1 = l$  are satisfied

<sup>8</sup>In the original statement of this theorem, the condition given is that ‘ $\text{ad} \bar{r}'(\text{Gal}(\overline{F}/F(\zeta_l)))$  is big’. While the notion of ‘big image’ is defined for a representation  $r$ , it is basically a property of the adjoint representation. Thus people often refer to  $\text{ad} \bar{r}$  as being big when they mean  $\bar{r}$  is big. I will try to consistently use the  $\bar{r}$  notation in this paper however.



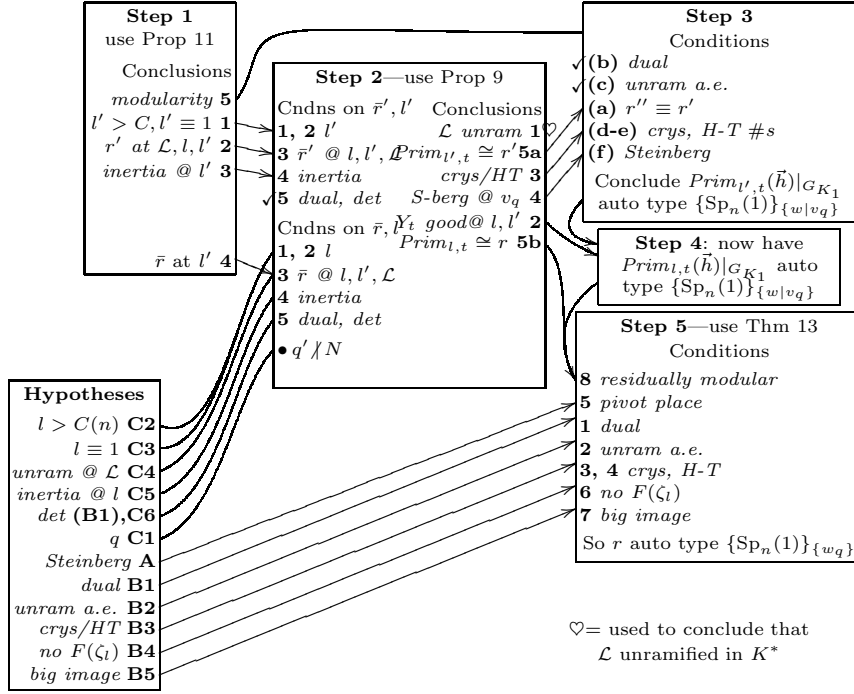


FIGURE 6. Logical structure of argument for Theorem 12

by hypotheses C2, C3, C4, C5 respectively, together with conclusion 4 of step 1 which controls  $\bar{r}$  at  $l'$ . Next, I claim that the determinant of the polarization on  $\text{Prim}[l]_v$  matches the determinant of the polarization on  $r$ ; this is from hypothesis B1, and the fact that  $l$  splits in  $F^*(n, N)$  which tells us  $\text{Prim}[l]_v$  has polarization with determinant a square by Proposition 10. Finally, we can use condition C6 to get the rest of condition 5.

Conditions 1 and 2 on  $l_2 = l'$  are satisfied by conclusion 1 of Proposition 11 applied in step 1; and conditions 3 and 4 on  $\bar{\rho}_2 = \bar{r}'$  are met respectively by conclusions 2, 3 of the same proposition. Finally, condition 5 on  $\bar{r}'$  is met since  $r'$  is symplectic with multiplier  $\epsilon^{1-n}$  (note that this automatically means that the determinant of the polarization will be  $-1$ , which is a square since  $l' \equiv 1 \pmod{4}$ ; this will match  $\text{Prim}[l']$  since  $l'$  splits in  $F^*(n, N)$ ).

We are left with a CM field  $K_1$ , a point  $t \in T_0(K_1)$ , and characters  $\chi_l$  and  $\chi_{l'}$  satisfying the conclusions 1–5 of Proposition 9.

*Step 3:* I claim that  $(\text{Prim}_{l',t}(\vec{h}) \otimes \chi_l)|_{G_{K_1}}$  is automorphic of weight 0 and type  $\{\text{Sp}_n(1)\}_{\{\Omega|v_q\}}$ . To check this, in the light of conclusion 5 of the Proposition in step 1, it suffices to check the conditions a–f given there. Conditions (a) and (f) are met by conclusions 5, 4 of the proposition in step 2, and conditions (d) and (e) are met by conclusion 3. Condition (b) is a simple geometric fact about our family established in Proposition 4 (point (2)). Finally, condition (c) is automatic since  $\text{Prim}_{l',t}$  is a piece of the cohomology of a variety and  $\chi_l$  is finite order.

We can immediately deduce that  $\text{Prim}_{l',t}(\vec{h})|_{G_{K_1}}$  itself is automorphic.

*Step 4:* Since  $Prim_{l',t}$  and  $Prim_{l,t}$  are part of a compatible system, which are crystalline/unramified (as appropriate) at  $l$  and  $l'$  (because of conclusion 2 of the proposition applied in step 2), the fact that  $Prim_{l',t}(\vec{h})|_{G_{K_1}}$  is automorphic implies  $Prim_{l,t}(\vec{h})|_{G_{K_1}}$  is also automorphic (of weight 0 and type  $\{Sp_n(1)\}_{\{\Omega|v_q\}}$ ).

*Step 5:* I claim that  $r|_{G_{K_1}}$ , is modular of weight 0 and type  $\{Sp_n(1)\}_{\{w_q\}}$ . We shall see this using Theorem 13. (Note that in applying this theorem we use the fact that  $l > n$ .) Conditions 1 and 2 are met by hypotheses B1, B2 respectively. Conditions 3 and 4 are both satisfied by condition B3, with  $a = 0$ . For condition 5, hypothesis A (and the fact  $w_q|v_q$ ) gives us what we need. Conditions 6 and 7 are met by hypotheses B4, B5 respectively. (For condition 7, we also use the fact that the field extension we made in step 2 was linearly disjoint from the fixed field of the kernel of  $\bar{r}$ .) Condition 8 comes from the fact that  $(Prim_{l,t'}(\vec{h}) \otimes \chi'_l)|_{G_{K_1}} \equiv r \pmod{l}$ .

This completes the proof of Theorem 12.  $\square$

#### REFERENCES

- [1] L. Clozel, M. Harris and R. Taylor, *Automorphy for some  $l$ -adic lifts of automorphic mod  $l$  representations*, to appear.
- [2] P. Deligne, J. S. Milne, A. Ogus, K.-Y. Shih, *Hodge cycles, motives and Shimura varieties*, LNM 900, Springer 1982.
- [3] R. Guralnick, M. Harris and N. M. Katz, *Automorphic Realization of residual Galois representations*, preprint.
- [4] M. Harris, N. Shepherd-Barron and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy*, to appear.
- [5] N. Katz, *Another look at the Dwork family*, to appear in Manin Festschrift.
- [6] N. Katz, *Exponential sums and differential equations*, Annals of Math. Study 125, Princeton Univ. Press, 1990.
- [7] C. R. Matthews, L. N. Vaserstein, and B. Weisfeiler, *Congruence properties of Zariski dense subgroups I*, Proc. Lon. Math. Soc. 48 (1984), 514–532.
- [8] M. Nori, *On subgroups of  $GL_n(\mathbb{F}_p)$* , Invent. Math. 88 (1987), 257–275.
- [9] R. Taylor, *Remarks on a conjecture of Fontaine and Mazur*, Journal of the Institute of Mathematics of Jussieu 1 (2002), 1–19.
- [10] R. Taylor, *Automorphy for some  $l$ -adic lifts of automorphic mod  $l$  representations, II*, to appear.

*E-mail address:* `tbl@math.harvard.edu`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA